

MISCELLANIES.

OR A

MISCELLANEOUS TREATISE;

CONTAINING

SEVERAL MATHEMATICAL

SUBJECTS.

by William Emerson

— Non ego paucis
Offendor Maculis, —

Hog.

L O N D O N :

Printed for J. NOURSE, Bookseller in ORDINARY to his
MAJESTY, in the Strand.

M DCC LXXVI.

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P R E F A C E.

TH E following Book consists of several Articles, in divers parts of the Mathematics. 1. The Laws of Chance; the principles of which are delivered in a short method, and is preparatory to what follows. 2. Of finding the values of Annuities on Lives; here the principal Problems concerning annuities and reversioners are solved. 3. The instituting or forming Societies; a thing much in vogue now. Here several of the cases are calculated upon true principles; by which it will appear, that hardly any of these Societies now in use can stand for any length of time upon the principles they have been calculated. In these three Articles, to avoid long calculations, I have frequently used approximations, which in most cases may serve well enough, for such an inconstant subject as Chance. 4. Of the Moon's Motion, where several of her irregularities are calculated. 5. The Construction of Arches; here are laid down several methods of performing this. 6. The Precession of the Equinoxes; which is universally solved upon all suppositions. 7. The Construction of Logarithms, by the roots or indices of powers. 8. Interpolation of Quantities, the best methods of doing this. 9. Of finding the Longitude at Sea; this method is the same as that which I gave a short account of in page 364, of my Astronomy; which took up but a single page; and yet I got myself abused for it, by an anonymous scribbler. I never knew before, that it was any crime to seek for new methods for solving difficult Problems. If I had failed in it, there was but one page of the Book lost. But here I have improved the method, and compleated

THE PREFACE.

it; and have shewn, that it will answer the purpose as near as there can be any occasion for. Indeed, I have constructed no Tables to ease the calculation. For though it requires a little more time, I should sooner chuse to go along the common high road, rather than go over hedge and ditch for the sake of a little gain. And this method I design only to be used occasionally; as the common known methods are sufficient in moderate weather. For certainly no body can be so silly as to think, that any methods by the Moon ought to be practised every day. These that are reckoned the best methods, have so many calculations attending them, that there is infinite danger of making mistakes, and therefore ought to be used sparingly.

10. The computation of Interest upon all manner of suppositions. 11. The Figures of Sines, Tangents, &c. and several things depending thereon. 12. Fortification, a short tract, laying down the general rules. 13. Gunnery, shewing the principles of that art. 14. Architecture, or the Art of Building, containing the rules and practice of this art. 15. Music, the nature and principles laid down. 16. A general Method of Philosophising, or rules to be observed in Philosophical Enquiries, with Examples. 17. Optics; I have here translated several papers of Sir J. Newton's Optical Lectures, Part II. which never was translated into English. The Phenomenon of the Rainbow is here computed in a way something different from what it is in his Optics, being here illustrated and wrought out in numbers, and therefore will be more intelligible. I wonder this latter part of the Optical Lectures has never been translated. For tho' the substance of it is contained in the Optics, yet here are a great number of curious experiments not mentioned there, which deserve to be made public. 18. A collection of curious, useful, and important Problems, many of them new; and those that are not, have new Solutions.

I have

THE PREFACE.

I have done all I could to keep clear of errors, but among so many things, some may possibly escape, for I have no pretence to infallibility; and if my Readers will not excuse me in this, they must not read my Book.

But I cannot dismiss this affair, without taking notice of a certain obscure Critic, who insulted and abused me in the Gentleman's Magazine, for some trifling numerical errors in the Astronomy, that any body, with the least smattering of Mathematics, might rectify. Yet this Drawcansir kept barking at me out of his hole, in several of the Magazines, and pretended to criticize my Book, though he acknowledged he did not understand it. A fine sort of a Critic indeed, to rail at me for his own dulness. Yet this doughty Author, when he came to be unkennelled, proved to be no more than a little paltry Schoolmaster, who did not so much as understand his own native Language; nor (by report) scarce ever writ a line of Mathematics in his life. This mighty Hero, in the last paper (which his trusty friend, the Compiler of the Magazine, put in for him) has inserted no less than a score of falsehoods, either directly such, or strongly insinuated. Yet this same impartial Compiler refused to do me the justice to put in my Remarks, which exposed these enormous lies, because his Friend and Favourite was going to give up. So that my paper is still in the hands of Mr. Nourie, where any body may see it that pleases.

Therefore my Readers may please to take notice, that if any envious, abusive, dirty Scribbler, shall hereafter take it in his head to creep into a hole like an Assassin, and lie lurking there on purpose to scandalize and rail at me; and dare not shew his face like a Man; I shall give myself no manner of trouble about such an Animal, but look upon him as even below contempt.

W. EMERSON.

POST-

P O S T S C R I P T.

Since Mayer's Tables came into my hands, I had a mind to try their exactness, by comparing them with Dr. Halley's Observations. Accordingly I found them to come very near in most cases, but not in all. For in many cases Dr. Halley's Tables come nearer; and so do some other Tables that I tried. And in making these comparisons here and there, I found sometimes an error of 2 minutes or more, and several not much short of 2 minutes. For example, Dec. 31st, 1725, the error is $-2' 45''$. Jan. 31st, 1726, the error is $+2' 21''$. And Apr. 15th, 1728, the error is $+2' 3''$. So that it may be truly said, that no Tables extant can be depended upon to give the Moon's place true to two minutes; contrary to the confident assertions of our little A, B, C, Critic, who has been so hot upon the matter.

And here I call upon this fiery Zealot, this boasting, upstart, nominal Astronomer, that among the many strange positions he has laid down, he will explain to the public the following particulars.

1. How he can make it intelligible, or even shew its possibility; that in taking the distance of two objects, it is only necessary to observe or look at one of them.
2. How he can make it appear, that the depriving a luminous body of its light, is an absurdity.
3. That he will point out these Corollaries (being near half in the Book of Astronomy) that have no dependance upon the Propositions they are placed under.
4. How he makes it out, that I have calumniated and condemned Mayer's Works; and wilfully and wickedly calumniated the Nautical Almanack. And particularly to shew what actions I have committed, which he says are worse than robbing the Widow and Fatherless.

5. How

5. *How it comes to be a fundamental principle in Philosophy, to ascribe the properties of a solid body to a mere image.*

6. *That, for the public good, and to keep men out of error, he will acquaint them with, and expose, the several absurdities he has lately found, and is always finding in my Book.*

7. *To assign a reason why the Eclipses of Jupiter's Satellites are not visible till long after the time expected, perhaps a quarter of an hour.*

8. *By what authority he perverts the meaning of my words, and imposes his own sense upon them, directly contrary to mine.*

This astronomical Conjuror will easily find his way into the Magazine; and when he has fully explained and cleared up all these Articles, men will be able to form a judgment of his mathematical abilities, as well as of his moral character, and, doubtless, the impartial world will judge accordingly.

W. E.

C O N.

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A R T.

A R T. I.

T H E

LAWS OF CHANCE.

THOUGH Chance may seem a fickle and ludicrous subject to treat upon, because there is no certainty or constancy in it. Yet all the events that happen have some certain degrees of probability for their happening or not happening; and these degrees of probability come under a mathematical consideration. Although it is impossible to determine with certainty how an event shall happen, yet it may be determined mathematically, what likelihood or degree of probability there is for its happening or failing; and this is all that is intended by the calculation. For scarce any thing falls out as it is determined by calculation, except there be made an infinite number of repetitions, and then one with another will always bring it to the same thing as the calculation makes it.

The Doctrine of Chances, therefore will be very useful to such persons as engage in play; for by this means they may know what advantage or disadvantage they have on their sides; and how to manage things to the best advantage in the play.

It is likewise a fit subject for the exercise of reason, for many problems about Chance; though they seem to be very simple, are attended with a long train of consequences, before we come at the

LAWS OF CHANCE.

conclusion, which is often quite different from what one would expect. And thus it corrects such mistakes as we are apt to make in our judgment.

DEFINITION I.

Chance is an event, or something that happens without the design or direction of any agent; and is directed or brought about by nothing but the laws of nature.

Therefore Chance has a natural, but not a final cause. It is the very nature of Chance to be inconstant.

D E F. II.

The *probability* or *improbability* of an event happening, is the judgment we form of it, by comparing the number of Chances there are for its happening, with the number of Chances for its failing.

D E F. III.

Expectation in play, is the value of a man's Chance; that is, of the thing played for, considered with the probability of gaining it; and therefore is the product of its value multiplied by the probability of obtaining the prize.

And the *advantage* = *expectation* — the *stake*.

D E F. IV.

Risk is the value of the stake considered with the probability of losing it; and therefore is the product of its value multiplied by the probability of losing it.

D E F. V.

Events are *independent* when they have no manner of connection with one another; or when the happening of one neither forwards nor obstructs the happening of any other of them.

Thus

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Thus if A undertake to throw an ace at two throws with one die; the first throw he has 1 Chance to win and 5 to lose; and the second throw he has also the very same, viz. 1 chance to win and 5 to lose; and the like as often as he pleases, and therefore these events are independent.

DEF. VI.

An event is *dependent* when the probability of its happening is altered by the happening of some other.

Thus if A undertakes to draw the ace out of 13 cards of a suit, at two trials or more. At the first trial he has 1 Chance to win and 12 to lose; at the second trial, a card being now taken away, he has 1 Chance to win and 11 to lose; and at the third, 1 Chance to win, and 10 to lose, &c. therefore these events are dependent.

AXIOM I.

In computing the number of Chances, it is supposed that all Chances are equal, or made with equal facility.

AXIOM II.

The whole expectation for any prize, is the sum of all the expectations upon the particulars.

AXIOM III.

The value of any Chance or expectation is what would purchase the like Chance or expectation, in a fair game.

PROP. I.

Putting 1 for absolute certainty, a for the number of cases (or chances) for the happening of an event.

B 2

b for

LAWS OF CHANCE.

b for the number of cases for its failing. Then $\frac{a}{a+b}$ is the probability of its happening; and $\frac{b}{a+b}$ is the probability of its failing: and the probabilities of happening and failing are as a to b .

For suppose a and b to be equal; no reason can be given why such an event will not as often fail as happen. And therefore the probability will be equal for both. And if a be greater or lesser than b , the probability will accordingly be greater or less for happening than for failing, and that in the ratio of a to b , or of $\frac{a}{a+b}$ to $\frac{b}{a+b}$.

Cor. The probability subtracted from 1, gives the probability of failing.

For $1 - \frac{a}{a+b} = \frac{a+b-a}{a+b} = \frac{b}{a+b}$ the probability of failing. And here 1 represents the sum of all the cases or chances, for and against happening. For it is a certainty, that some out of the whole must happen.

EXAMPLE.

If I undertake to throw an ace with one die; it is plain I have but 1 chance out of 6 for its happening; and 5 out of 6 for its failing; therefore my chance is $\frac{1}{1+5}$ or $\frac{1}{6}$, being the probability of its happening; and $\frac{5}{6}$ is the probability of its failing.

SCHOLIUM.

As 1 represents a certainty, or when an event has an infinitely great probability of happening. So $\frac{1}{2}$ will represent an equal probability for happening or failing. For $1 - \frac{1}{2} = \frac{1}{2}$.

P R O P.

PROP. II.

If a be the number of chances for happening, and b the number of chances for the failing, of one event.

And c the number of chances for happening, and d the number of chances for failing of another event independent of the former. Then

The product $\overline{a+b} \times \overline{c+d}$ or $ac+ad+bc+bd$ will be the number of all the chances, or of all the possible cases of the happening and failing of these events.

For $a+b$ denotes all the chances in the first event, and since $a+b$ may be connected with every one in $c+d$, therefore the sum of all the cases will be $\overline{a+b} \times \overline{c+d}$.

Cor. 1. By the same reasoning, if a, c, e , &c. be the chances for happening; and b, d, f , &c. those for failing, in 3 or more independent events. Then $\overline{a+b} \times \overline{c+d} \times \overline{e+f}$, &c. will be the number of all the chances or possible cases for all these events, to happen or fail.

Cor. 2. If there be proposed two events, the number of cases for the happening of the first, is $ac+ad$; for its failing, $bc+bd$. That both shall happen, is ac ; for the contrary, $ad+bc+bd$. That one shall happen, is $ac+ad+bc$; for both failing, bd ; that the first shall happen and $\} ad$; for the contrary, $ac+bc+bd$. the latter fail, is $\}$ and so of others.

Cor. 3. And by the same way of reasoning, if three or more events be proposed; the number of cases of their happening or failing, in any manner, will be had by multiplication only. So that by taking one or more terms according to the nature of the question, any problem about chance will easily be answered.

PROP. III.

The probability of happening of several independent events $\left(\frac{a}{a+b}, \frac{c}{c+d}, \frac{e}{e+f}, \text{ \&c.}\right)$ is the product of the several probabilities taken singly, $\frac{ace}{a+b \times c+d \times e+f}$.

For in one event, $\frac{a}{a+b}$ is the probability of happening, by Prop. I. and in two events, where the probabilities are $\frac{a}{a+b}$ and $\frac{c}{c+d}$; $a+b \times c+d$ is the number of all the cases or chances, and ac the number of cases where both events may happen; therefore $\frac{ac}{a+b \times c+d}$ = probability of both happening. In like manner $\frac{ace}{a+b \times c+d \times e+f}$ = probability of three events happening. And the Prop. holds in respect of failing, as well as of happening of events.

Cor. 1. If $x, y, z, \text{ \&c.}$ denote the probabilities of several events happening as A, B, C, \&c. then $1-x, 1-y, 1-z, \text{ \&c.}$ will be the probabilities of their failing. Then

xyz = probability of their all happening.

$1-x \times 1-y \times 1-z$ = probability of their all failing.

$1-1-x \times 1-y \times 1-z$ = probability of some of them happening.

$xy \times 1-z$ = probability of happening of A, B, and failing of C.

And so of others.

Cor.

COR. 2. *What is said about the probability of happening, is equally true about the probability of failing, viz. that it is equal to the product of the probabilities of failing of the several particulars.*

COR. 3. *To determine the probability of happening of several dependent events; distinguish them into first, second, third, &c. Then the probability of the first must be looked upon as independent, the probability of the second's happening, must be determined on the supposition of the first having happened some way or other; and the probability of the third's happening, is to be determined from the supposition of the first and second having likewise happened; and so on. And then the probability of the happening of them all will be the product of the several probabilities as thus determined.*

PROP. IV.

If a be the number of chances for happening, and b for the failing of an event; the probability of happening at least t times in n trials will be

$$a^n + na^{n-1}b + \frac{n \cdot n-1}{2} a^{n-2}b^2 + \&c. \text{ till } a^t$$

$$= \frac{\quad}{a+b^n}, \text{ that}$$

is, till the index of a be t .

For by Prop. II. and its corollaries, if a , b , and c , d , and e , f , &c. be the chances for happening and failing of 3 or more events A, C, E; then the terms (of the multiplication) where a is concerned, will be the numerator of a fraction, shewing the probability of the first event happening. And the terms where a and c are concerned, give the numerator of a fraction where the first and second events both happen, and so on; and whose denominator is $a+b \times c+d \times \&c$.

B 4

Now

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Now let $c+d, e+f, \&c.$ be the same event as $a+b$, and then $\frac{a+b \times c+d \times e+f, \&c.}{a+b \times a+b \&c.} = \frac{a+b^n}{a+b^n}$; and $c, e, \&c.$ being $= a$; the happening of a and c is equivalent to the happening of a twice, and the happening of a, c, e ; is equivalent to the happening of a thrice, and so on. And therefore the terms with a relate to the happening of the event once, and where aa is concerned, relate to happening twice; and a^3 to happening thrice. Therefore all terms with a as far as a^t , must be taken into the numerator for happening t times, and the denominator will be $a+b^n$; and the fraction, shewing the probability of happening t times, will be that above laid down.

Cor. 1. If a and b be the chances for happening and failing of an event; then $1 - \frac{b^n}{a+b^n}$ is the probability of its happening, and $\frac{b^n}{a+b^n}$ the probability of its failing once in n trials.

For b^n is the only term in which a is not found, and therefore $\frac{b^n}{a+b^n}$ is the probability of its failing, which taken from 1 gives the probability of happening.

Cor. 2. The probability of happening precisely t times in n trials is $\frac{T}{a+b^n}$, where T is that term, where the index of a is t .

PROP. V.

If a person has an equal chance for the two prizes A, B ; the value of his expectation is $\frac{A+B}{2}$.

For

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For since there is no reason for one chance happening sooner than another, in an equality of chances; therefore (by Prop. I.) the probability of the happening of each of the two is $\frac{1}{2}$; and therefore the expectation for A is $\frac{1}{2} A$; and the expectation for B is likewise $\frac{1}{2} B$. And if I would purchase the whole, I must give the value, which is $\frac{1}{2} \times A + B$.

For the same reason, the probability of taking any one of 3 things A, B, C, (by equality of chances) is $\frac{1}{3}$ (by Prop. I.); whence the expectation of A is $\frac{1}{3} A$, and of B, is $\frac{1}{3} B$, and of C, $\frac{1}{3} C$; and the whole value = $\frac{1}{3} \times A + B + C$, whence this

Cor. If there be an equal chance for 3 prizes A, B, C; the expectation = $\frac{A+B+C}{3}$. And for 4 things A, B, C, D; the expectation is $\frac{A+B+C+D}{4}$; and so on.

Schol. The two last propositions are very serviceable for solving many problems of Chance.

P R O P. VI.

If there be a chance for the prize A, and b chances for the prize B; and all the chances happen with equal ease; then the value of the expectation is worth $\frac{aA+bB}{a+b}$.

For (by Prop. I.) the probability of the happening of any chance for A is $\frac{a}{a+b}$; and for B, is $\frac{b}{a+b}$. Therefore the expectation for A is $\frac{aA}{a+b}$; and

and for B is $\frac{bB}{a+b}$. Therefore if I would purchase these expectations, I must give the value of the whole, (by Ax. 2.) which is $\frac{aA+bB}{a+b}$.

And by the same reasoning, if there be 3 prizes A, B, C; and the chances for each be a, b, c , respectively. Then the probability for A is $\frac{a}{a+b+c}$; and for B, $\frac{b}{a+b+c}$; and for C, $\frac{c}{a+b+c}$. And the expectation for A = $\frac{aA}{a+b+c}$, for B = $\frac{bB}{a+b+c}$, for C = $\frac{cC}{a+b+c}$. Whence this

Cor. If a, b, c , be the chances for A, B, C, respectively; the value of the expectation = $\frac{aA+bB+cC}{a+b+c}$; and so on.

P R O P. VII.

If a and b be two numbers of particular cases relating to A and B. And p be some number of chances, alike related to a and b . Then the share of p belonging to A is $\frac{a}{a+b}p$; and B's share $\frac{b}{a+b}p$.

For since p is promiscuously and alike related to the whole a and b ; it must be divided between them, in proportion to the numbers a and b . And the greater number must have the greater share, and the lesser number the lesser. Therefore if $a+b=s$, it will be as $s : p :: a : \frac{pa}{s} = \text{A's share}$; and $s : p :: b : \frac{pb}{s} = \text{B's share}$.

Cor.

ART. I. LAWS OF CHANCE.

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Cor. If there be several subjects A, B, C, D, &c. whose numbers are $a, b, c, d, \&c.$ And the number of chances or things p be alike related to them all. Then the shares of A, B, C, D, &c. are respectively $\frac{ap}{s}, \frac{bp}{s}, \frac{cp}{s}, \frac{dp}{s}, \&c.$

SCHOL.

What is meant here by A, B, C, D, &c. is any species of beings; and by $a, b, c, d,$ the like species of beings under particular circumstances; and which may belong equally to any of them.

PROP. VIII.

If there be n equal chances in all, and 1 for an event given; that event will happen once in n trials, taking one time with another.

For taking an infinite number of trials, that event in the whole will happen once in every n trials, because all chances fall out with equal facility or difficulty. And consequently the more trials the more it will approach to that ratio. And in an infinite number of repetitions, it approaches infinitely near a certainty, to fall out once in n trials. And therefore at a medium we ought to reckon, that this one chance will fall out in n trials.

Cor. 1. If there be n equal chances in all; it is an even wager that a given chance will fall out in $\frac{1}{2} n$ trials. Or, there is an equal probability for its happening or failing in $\frac{1}{2} n$ trials, taking one time with another.

For in a great number of repetitions of n trials, one chance will fall out for every repetition of n trials, or in every 2 repetitions of $\frac{1}{2} n$ trials. Therefore in these 2 repetitions one will happen and one will

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will fail continually; and therefore it becomes equally probable, whether it will happen or fail in $\frac{1}{2}n$ trials.

Cor. 2. Hence, to know how any event will fall out at a medium; it will be proper to consider what the result will be, by an infinite number of trials.

For innumerable trials take away all irregularities, and set all things in due proportion.

P R O P. IX. PROB.

If a be the chances or number of cases for happening, and b the number for the failing of an event. To find at how many trials, it is equally probable to happen or fail.

Let x be the number of trials. Then (Cor. 1. Prop. IV.) $1 - \frac{bx}{a+b^x}$ = probability of happening,

which therefore is $= \frac{1}{2}$, whence $\overline{a+b^x} = 2b^x$. And $x \times \log. \overline{a+b} = \log. 2 + x \times \log. b$; therefore $x \times \log. \overline{a+b} - x \times \log. b = \log. 2 = .30103$,

$$\text{and } x = \frac{.30103}{\log. \overline{a+b} - \log. b} = \frac{.30103}{\log. : \frac{a+b}{b}}.$$

Cor. 1. If $a=b$, then $x=1$.

For $\log. \frac{a+b}{b} = \log. \frac{2a}{a} = \log. 2 = .30103$,

$$\text{and therefore } x = \frac{.30103}{.30103} = 1.$$

Cor. 2. If b is very great in respect of a , then $x = \frac{7}{10} \times \frac{b}{a} = .7 \frac{b}{a}$, nearly.

For

For $\log. \frac{a+b}{b} = \log. 1 + \frac{a}{b}$; where $\frac{a}{b}$ is a very small quantity; and therefore by the nature of logarithms (Alg. Prob. 84. Cor. 1.) putting $v = \frac{a}{b}$, $\log. 1 + v = .4343 \times v - \frac{vv}{2}$, &c. that is,

$$\log. 1 + \frac{a}{b} = .4343 \times \frac{a}{b}, \text{ \&c. therefore } x =$$

$$\frac{.30103}{\log. 1 + \frac{a}{b}} = \frac{.30103}{.4343 \times \frac{a}{b}} = \frac{.30103}{.4343} \times \frac{b}{a} = .693 \times$$

$$\frac{b}{a} = .7 \times \frac{b}{a} \text{ nearly.}$$

Cor. 3. If b is very small or 0, then x becomes very small or nothing.

Thus if $1 + \frac{a}{b} = 10, 100, 1000, 10,000, \&c.$
 then will $x = .3, \frac{.3}{2}, \frac{.3}{3}, \frac{.3}{4}, \&c.$ respectively.

PROP. X. PROB.

If there be a chances for happening, and b chances for failing of an event at 1 trial; to find in how many trials it is equally probable or improbable for happening t times.

If the trials be continued *ad infinitum*, the event will happen a times, and fail b times in every $a+b$ trials. And therefore one time with another it will happen once in every $\frac{a+b}{a}$ trials, and consequently

$\frac{t}{t-1}$ times in $\frac{t}{t-1} \times \frac{a+b}{a}$ trials. To this add the

number of trials, in which there is an equal chance for happening once, which (by Cor. 2. Prop. IX.) is $\frac{.7b}{a}$, when b is large, and the number of trials

for

for happening n times will be $\frac{nb}{a} + t-1 \times \frac{a+b}{a}$
 $= t - \frac{3}{10} \times \frac{b}{a} + t-1.$

Therefore in $t - \frac{3}{10} \times \frac{b}{a} + t-1$ trials, it is an equal chance whether the event will happen t times or not. But if a and b be nearly equal, the number of trials will be $\frac{tb}{a} + t-1.$

This method is the more exact as b is greater.

Otherwise, more exactly.

Put x = number of trials, then (by Prop. IV.) we

$$a^x + xa^{x-1}b + x \cdot \frac{x-1}{2} a^{x-2}bb + \&c.$$

shall have $\frac{2}{a+b} = \frac{1}{x};$

the series to be continued till the index of a be equal to t . Or else make

$$\frac{b^x + xb^{x-1}a + x \cdot \frac{x-1}{2} b^{x-2}aa + \&c.}{a+b}$$

$= \frac{1}{x},$ continuing the series till the index of a be $t-1$; and from either of them find the value of x . The former to be used when t is great, and the latter when t is small. But when x is a large number, this will prove a very difficult calculation.

PROP. XI.

If A and B play together, and A has a chances to win, and B has b chances. The probability of A's winning, is to the probability of B's winning, as a to b .

This is the foundation of all gaming; for the more chances the greater the probability of winning; and the greater the probability the greater the success, taking one time with another. Now the

the probability of A's winning is $\frac{a}{a+b}$, and that of B's winning, $\frac{b}{a+b}$; and these are as a to b .

Cor. 1. Hence the odds in any wager laid upon a game, should be as the chances; that is, the money laid on A, to the money on B, should be as a to b , in a fair game. And the stakes should also be in the same ratio.

Cor. 2. Hence also, one time with another, A will win a games, for B's winning b games.

P R O P. XII.

To resolve a Problem by the Laws of Chance.

1. The first thing to be done in any problem relative to chance, is to find out how many cases there are in which the event will happen or fail; and likewise the number of all the cases possible. So you will have the number of chances for and against you; this gives the probability of the event. But great care must be taken to allow no more nor fewer chances for an event, than really there are; for it is a frequent error to allow too many.

2. Oftentimes the probability of the failing of an event, or the number of cases that fail, may be more easily found, than those for the happening. Therefore find these first, and from thence those for the happening will be easily found.

3. When several conditions are contained in the problem, each of which requires its proper degree of probability; then these several probabilities must be separately found, and all added together for the total probability.

4. In

4. In many complicated problems, there will be so many cases for the happening and failing of an event; that it will be a very tedious piece of work to count them up. In such cases, recourse must be had to the doctrine of combinations and permutations, to obtain a complete solution.

5. Many problems require the help of Prop. V. and VI, especially for two things A, B. In applying them, we must find the expectation, both when the event happens and fails, and take the sum as the Prop. directs; and so proceed from the simple cases to the harder.

S C H O L I U M.

Having laid down the foregoing propositions, containing the foundation of the Doctrine of Chances; I shall now give a few problems, to shew the use and application thereof in some easy cases, as an introduction to that art; referring the reader to that excellent treatise of M. de Moivre's, for the determination of problems of greater difficulty.

P R O B. I.

If I have six equal chances for 2, 3, 5, 8, 10, and 11 shillings (and 4 chances for nothing) what is the value of my expectation.

The probability of gaining each of these prizes, is $\frac{1}{10}$, therefore the sum of my expectations is $\frac{1}{10} \times 2+3+5+8+10+11 = \frac{39}{10} = 3.9$ shillings the value required.

Or thus, $\frac{2+3+5+8+10+11}{10} = \frac{39}{10} = 3.9$, by Prop. V.

P R O B.

P R O B. II.

If I have 3 chances for 4 shillings, 6 chances for 8 shillings, and 5 chances for 5 shillings; what is the value of my expectation, these 14 being all the chances possible.

By Prop. VI. A, B, C, are respectively equal to 4, 8, 5, shillings; and a, b, c , equal to 3, 6, 5. Therefore my expectation is worth $\frac{3 \times 4 + 6 \times 8 + 5 \times 5}{14}$

$$= \frac{12 + 48 + 25}{14} = \frac{85}{14} = 6\frac{1}{14} \text{ shillings.}$$

P R O B. III.

To find the probability of casting an ace in three throws with one die, or at one throw with three dice.

By Prop. VIII. the ace will be cast in 6 throws, taking one time with another; and in 3 throws it will be an equal chance whether it be cast or not. According to this the probability of casting it in 3 throws is $\frac{1}{8}$, nearly.

Otherwise. By Prop. IV. put $a=1, b=5, n=3, t=1$; then $\frac{a^3 + 3a^2b + 3abb^2}{a+b^3} = \text{probability; that is,}$

$$\frac{1+15+75}{216} = \frac{91}{216}.$$

Or, $\frac{b^3}{a+b^3} = \frac{125}{216} = \text{probability of failing; and}$
 therefore $1 - \frac{125}{216} = \frac{91}{216}$ is the probability of happening, as before.

C

P R O B.

P R O B. IV.

To find the probability of casting an ace in 6 throws with one die.

Though this event will happen in 6 trials, taking one time with another (by Prop. VIII.); yet in barely 6 trials it cannot be depended on. For in an infinite number of repetitions it continually approximates to a certainty of throwing an ace in 6 throws; yet in one repetition only, it has but a certain degree of probability. To find which,

Put $a=1$, $b=5$, $n=6$, $t=1$. Then (Prop. IV.)
 $\frac{b^6}{n^6}$ = probability of failing = $\frac{5^6}{6^6} = \frac{15625}{46656}$, and the
 probability of happening = $1 - \frac{15625}{46656} = \frac{31031}{46656}$
 $= \frac{2}{3}$ nearly;

Otherwise.

Let 1^{st} be the prize. At the 1st throw there is 1 chance for me and 5 against me, (and the like every throw). Therefore (by Prop. VI.) I have 1 chance for 1^{st} , and 5 for 0; therefore my expectation = $\frac{1 \times 1 + 5 \times 0}{6} = \frac{1}{6}$, for 1 throw.

For 2 throws. If I hit the first time I gain 1, if I miss it, I have 1 throw remaining = $\frac{1}{6}$. There-

fore I have 1 chance for 1, and 5 for $\frac{1}{6}$, and my

$$\text{expectation} = \frac{1 + 5 \times \frac{1}{6}}{6} = \frac{11}{36}.$$

For

For 3 throws; my expectation $\frac{1+5 \times \frac{1}{6}}{6} = \frac{91}{216}$,
as in Prob. III.

Proceeding thus to 6 throws, we get $\frac{1+5 \times \frac{4651}{7776}}{6}$
 $= \frac{31031}{46656}$, as before.

P R O B. V.

*In how many throws may one undertake for an even
wager to cast an ace with one die?*

Let x = number of throws, $a = 1$ = chance to
hit, $b = 5$ = chances to miss. Then (Cor. 1. Pr.
IV.) $\frac{b^x}{a+b^x}$ = probability of failing, which (in an
even wager) is equal to the probability of happen-
ing; therefore $\frac{b^x}{a+b^x} = \frac{1}{2}$, and $2b^x = a+b^x$.

Whence $\log. 2 + \log. b^x = \log. a+b^x$, or
 $\log. 2 + x \times \log. b = x \times \log. a+b$; and $x \times \log.$
 $a+b - x \times \log. b = \log. 2$. Therefore $x =$
 $\frac{\log. 2}{\log. a+b - \log. b} = \frac{\log. 2}{\log. 6 - \log. 5} = \frac{.30103}{.07918}$
 $= 3.82$, that is, between 3 and 4 throws; or near-
ly at 4 throws, one may undertake to throw an ace.

Otherwise, proceed (by Prop. VI.) as in the last
example, till the expectation be $= \frac{1}{2}$.

P R O B. VI.

*In how many throws with one die, may one under-
take to throw an ace 4 times?*

C 2

Let

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Let x = number of throws, $a=1$, $b=5$, $t=4$, then proceeding as before, we shall have (by Prop. IV.)

$$\frac{a^x + x a^{x-1} b + x \cdot \frac{x-1}{2} a^{x-2} b b \&c.}{a + b^x} = \frac{1}{2}, \text{ or } a^x + x a^{x-1} b$$

$$+ x \cdot \frac{x-1}{2} a^{x-2} b b \&c. \text{ till } a^4 = \frac{1}{2} \times \overline{a + b^x} = \frac{a + b^x}{2}$$

To find x , the easiest way is to take it by guess, and add 1 every trial, which will increase a term of the series every time; proceed till you find the two sides of the equation nearly equal, and then you have the value of x nearly. But to shorten the work, proceed thus. Since an ace may be thrown every 6 throws, taking one time with another for a great many repetitions; and by Prob. V. at one turn only, (or a few turns) it will happen once in 3.82 throws; therefore the ace to be thrown 4 times, requires at least 4 times 3.82 throws, or 15.28 throws; but to be thrown at 6 throws always, requires 4 times 6, or 24 throws at most. Therefore x will probably be between 15 and 24. And therefore I shall suppose it, at a mean, to be

$$19. \text{ Then we have } a^{19} + 19 a^{18} b + 19 \cdot \frac{18}{2} a^{17} b b \&c.$$

$$= \frac{a + b^{19}}{2}. \text{ But since here will be a great number}$$

of terms before we come at the index a^4 ; therefore it will be shorter to take the other end of the series, for the failing; since in an equal wager, they are of equal value; and then we have $b^x + x b^{x-1} a$

$$+ x \cdot \frac{x-1}{2} b^{x-2} a^2 + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} b^{x-3} a^3 = \frac{a + b^x}{2} : \text{ and thus}$$

we have but 4 terms to add up. In the present supposition $5^{19} + 19 \times 5^{18} + 171 \times 5^{17} + 969 \times 5^{16}$

$= \frac{6^{19}}{2}$, where a is $=1$. These put into numbers by help of logarithms, will give 517, which should be 305, neglecting superfluous numbers or places.

Again, taking $x = 20$, we get the series 207, which should be 183, therefore x is still too little; therefore,

Take $x = 21$, then the series is 115, which is nearly $=109$, or $= \frac{6^{21}}{2}$. Again,

Take $x = 22$, and the series is $= 642$, which is something more than 656, the 22d power of 6 divided by 2; therefore here x is too great.

Whence we find x between 21 and 22, being very little more than 21, the number of throws required.

Cor. 1. *Hence in this example, the number of throws required is nearly so many times 6, abating one, $+3.8=18+3.8=21.8$. And probably is nearly the same in other like cases.*

As if it were required to find how many throws are required to throw an ace 6 times; the answer is $5 \times 6 + 3.8 = 33.8$, or near it.

Cor. 2. *And hence may be collected, at how many throws, the same may be done with several dice.*

As if it be required to find in how many throws one may undertake to throw 4 aces with 6 dice. Having found the number of throws with one die, 21; divide it by the number of dice; so $\frac{21}{6} = 3\frac{1}{2}$, the throws required with 6 dice.

SCHOLIUM.

Many questions in chance of this sort, require such an infinite deal of labour in calculating, that

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few people can have the patience to go through it. And therefore it is better in such cases to be content with such approximations as come near the matter, since chance is but chance; or, in other words, since chance affords nothing certain; hardly any thing ever falling out as it is calculated.

L E M M A.

To find the number of cases to throw a given number, with any number of dice.

Let the given number of points = $p + 1$,
 n = number of dice,

Then the number of cases is

$$\begin{aligned}
 &+ \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}, \text{ \&c. (to } n-1 \text{ terms)} \\
 &- \frac{p-6}{1} \times \frac{p-7}{2} \times \frac{p-8}{3}, \text{ \&c. } \times : \frac{n}{1} \\
 &+ \frac{p-12}{1} \times \frac{p-13}{2} \times \frac{p-14}{3} \times : \frac{n}{1} \times \frac{n-1}{2} \\
 &- \frac{p-18}{1} \times \frac{p-19}{2} \times \frac{p-20}{3} \times : \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \\
 &+ \text{ \&c.}
 \end{aligned}$$

Which rows of series must be continued downwards, till some of the factors in each become 0 or negative; taking $n-1$ factors in each.

This is demonstrated by Mr. D' Moivre, but I have no room for it in this place.

E X A M.

E X A M.

How many chances are there for throwing 16 with 4 dice.

Here $p=15$, $n=4$, whence

$$+ \frac{15}{1} \times \frac{14}{2} \times \frac{13}{3} = 455$$

$$- \frac{9}{1} \times \frac{8}{2} \times \frac{7}{3} \times \frac{4}{1} = -336$$

$$+ \frac{3}{1} \times \frac{2}{2} \times \frac{1}{3} \times \frac{4}{1} \times \frac{3}{2} = +6$$

$$\text{Sum} \quad \underline{\underline{125}}$$

And the whole number of chances is 6^4 or 1296.

P R O B. VII.

What is the probability of casting 15 points with 6 dice?

Here $p=14$, $n=6$, whence

$$+ \frac{14}{1} \times \frac{13}{2} \times \frac{12}{3} \times \frac{11}{4} \times \frac{10}{5} = +2002$$

$$- \frac{8}{1} \times \frac{7}{2} \times \frac{6}{3} \times \frac{5}{4} \times \frac{4}{5} \times 6 = -336$$

$$\text{Numb. points} \quad \underline{\underline{1666}}$$

But $6^6 = 46656$; therefore the probability is
 $\frac{1666}{46656}$

P R O B. VIII.

A undertakes to throw 10 with 3 dice, before B throws 12 with 4 dice. What is the odds of the game?

C 4

First,

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First for the probability of casting 10 with 3 dice; here $p = 9$, $n = 3$, then

$$+ \frac{9}{1} \times \frac{8}{2} = + 36$$

$$- \frac{3}{1} \times \frac{2}{2} \times \frac{3}{1} = - 9$$

$$\text{Number of cases} \quad \underline{\underline{27}}$$

But the whole number of cases is 6^3 or 216, therefore the probability at one throw is $\frac{27}{216}$ for A.

Secondly, for the probability of casting 12 with 4 dice. Here $p = 11$, $n = 4$, therefore

$$+ \frac{11}{1} \times \frac{10}{2} \times \frac{9}{3} = + 165$$

$$- \frac{5}{1} \times \frac{4}{2} \times \frac{3}{3} \times \frac{4}{1} = - 40$$

$$\text{Number of cases} \quad \underline{\underline{125}}$$

But here the whole number of cases is 6^4 , or 1296, therefore B's probability at one throw is $\frac{125}{1296}$. Whence A's chance is to B's as $\frac{27}{216}$ to $\frac{125}{1296}$, or as $\frac{27}{6^3}$ to $\frac{125}{6^4}$ or $\frac{162}{6^4}$ to $\frac{125}{6^4}$, that is, as 162 to 125; so that A has the better of the wager.

P R O B. IX.

Any number (n) of dice being given; to find the probability of casting 2 faces of a sort, at one throw.

In two dice, to avoid having 2 faces of a sort, the 6 faces of one die can only be combined with
5 faces

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5 faces of the other, and this number of combinations will be $6 \times 5 = 30$.

In three dice, the 30 throws of 2 dice, which have no two of a sort, can only be combined with 4 of the third die; for being combined with 5, there would be several times two of a sort. And the number of these combinations will be $6 \times 5 \times 4 = 120$.

Again, in 4 dice, to avoid 2 of a sort, the 120 of 3 dice can only be combined with 3 of the 4th die; and these combinations are $6 \times 5 \times 4 \times 3 = 360$, and so on.

Therefore there will be had the probability of failing; which will be, for 2 dice, $\frac{6 \times 5}{6 \times 6}$; for three dice $\frac{6 \times 5 \times 4}{6 \times 6 \times 6}$; for four dice $\frac{6 \times 5 \times 4 \times 3}{6 \times 6 \times 6 \times 6}$, &c. and univervally for n dice, $\frac{6 \times 5 \times 4 \text{ (to } n \text{ terms)}}{6 \times 6 \times 6 \text{ (to } n \text{ terms)}}$; because $6 \times 6 \times 6$ to n terms is the whole number of chances. Whence we have the probability of happening $= 1 - \frac{6.5.4}{6.6.6} \text{ (to } n \text{ terms.)}$

EXAMPLE I.

What is the probability of casting 2 faces of a sort with 4 dice.

Here $\frac{6.5.4.3}{6.6.6.6} = \frac{360}{1296}$ the probability of failing, and $\frac{1296 - 360}{1296} = \frac{936}{1296}$ the probability of happening.

EXAM. II.

What is the probability of throwing 2 faces of a sort with 6 dice.

Here

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Here $\frac{6.5.4.3.2.1}{6.6.6.6.6.6} = \frac{120}{7776}$ the probability of failing, and $\frac{7656}{7776}$ the probability of happening.

E X A M. III.

To find the probability of throwing two faces of a sort with 7 dice.

Here $\frac{6.5.4.3.2.1.0}{6.6.6.6.6.6.6} = \frac{0}{46656}$ the probability of failing, and $\frac{46656}{46656} = 1$ the probability of happening, which is a certainty.

Therefore in 7 dice or more, there must necessarily be two of a sort.

P R O B. X.

In how many throws may one undertake to throw all the faces of a die.

It is impossible to perform this in less than six throws. And taking one time with another, the old faces are as likely to come up as new ones, which will require twice the number at least. Therefore to throw all the faces, will require at least 12 casts.

Otherwise.

At the first throw some of the faces must necessarily be cast. At the 2d throw the probability of casting a different face is $\frac{5}{6}$. At the 3d throw, the probability of casting one of the remaining faces is $\frac{4}{6}$. And at the 4th, 5th, and 6th throws,
the

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the probability of still casting one of the remaining faces will be $\frac{3}{6}$, $\frac{2}{6}$, and $\frac{1}{6}$, respectively. Then to find the number of throws requisite for each.

The probability $\frac{5}{6}$ requires— $1\frac{1}{5}$ throws.

The probability $\frac{4}{6}$ requires— $1\frac{1}{2}$ throws.

The probability $\frac{3}{6}$ requires—2 throws.

The probability $\frac{2}{6}$ requires—3 throws.

The probability $\frac{1}{6}$ requires—6 throws.

in all	13.7
for the first add	1.0
	14.7

Therefore in 14 throws all the faces will probably be cast.

Mr. D' Moivre by a difficult calculation finds 13 throws for casting all the faces, which doubtless is more exact than mine, which are only so many approaches towards truth.

PROB. XI.

A person undertakes with 4 dice to throw 12, 13, 14, or 15 every throw. What is the odds?

By the Lemma (Prob. VII.) we shall find,

The chances for 12	—	125
The chances for 13	—	140
The chances for 14	—	146
The chances for 15	—	140
All the chances for him		551

All

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All the chances on 4 dice $= 4^6 = 1296$.

And $1296 - 551 = 745$ the chances against him.
Then the odds against him is as 745 to 551.

P R O B. XII.

To find the probability of casting m aces only, at n throws with one die, or at 1 throw with n dice.

Here n must be equal to, or greater than m . Let C = combinations of all the aces, taken by ones, or by twos or threes, &c. Then,

1. Suppose only one ace to be cast, then $m=1$, therefore setting aside one die, the remaining dice will be $n-1$; and to avoid aces with these, only 5 points of each must be taken. And the number of combinations of all these will be 5^{n-1} , which combined with all the aces in each single die, will be $C \times 5^{n-1}$, where one ace is cast.

2. Suppose $m=2$, then setting aside 2 dice, the number of combinations without any ace in $n-2$ dice, will be 5^{n-2} ; and these, combined with every 2 aces in the remaining dice, will be $C \times 5^{n-2}$.

3. If $m=3$, the combinations without any ace will be 5^{n-3} , and these combined with every three, will be $C \times 5^{n-3}$.

And in general, the combinations, without any ace, in $n-m$ dice will be 5^{n-m} ; and these combined with all the m 's out of n aces, will be $C \times 5^{n-m}$ for the number of cases that m aces can be cast, excluding all aces whatever, besides these.

Now by the doctrine of combinations, when $m=1$, $C=n$, and when $m=2$, $C = \frac{n \cdot n-1}{1 \cdot 2}$. And

when $m=3$, $C = \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3}$, and in general, for

m aces,

m aces, $C = \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4}$ (to m terms). And

since 6^n is the number of all the chances on n dice;

therefore $\frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3}$ (to m terms) $\times 5^{n-m}$, is the

number of all the cases that m aces only can be

cast, and $\frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3}$ (to m terms) $\times \frac{5^{n-m}}{6^n} =$ prob-

ability of casting them.

EXAMPLE.

What is the probability of casting 3 aces only, with 10 dice; or with 10 throws of 1 die.

Here $m=3$, $n=10$, and the number of chances is $\frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} \times 5^7 = 120 \times 78125 = 9375000$. And

$6^{10} = 60466176$; therefore the probability is =

$\frac{9375000}{60466176} = \frac{2}{13}$ nearly.

P R O B. XIII.

What is the probability of throwing both an ace and a six, with three dice.

By Prop. IV. the probability of casting an ace once in three throws of 1 die, or in one throw of

3 dice, is $1 - \frac{b^3}{a+b^3} = \frac{a+b^3-b^3}{a+b^3} = \frac{6^3-5^3}{6^3}$, where

$a=1$, $b=5$. And the number of chances = 6^3-5^3 .

Set aside the six, then for the same reason the number of chances for the ace is 5^3-4^3 .

Restore the six, and the chances for the ace to come up without the six, will be the same as if the six was wanting.

Now

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Now from the chances for the ace to come up, with or without the six ($6^3 - 5^3$), subtract the chances for the ace to come up without the six ($5^3 - 4^3$); there will remain the chances for both the ace and six to come up ($6^3 - 2 \times 5^3 + 4^3$). Consequently $\frac{6^3 - 2 \times 5^3 + 4^3}{6^3} = \frac{30}{216} = \frac{5}{36}$ is the probability required.

S C H O L.

By the same method $\frac{6^4 - 3 \times 5^4 + 3 \times 4^4 - 3^4}{6^4}$ will be the probability of casting 3 given faces with 4 dice.

P R O B. XIV.

A, B, C, play with two dice, the highest number take all; and A has cast 8. What is the value of his expectation for the prize S?

The mean number on two dice is 7, which must be reckoned for B and C, without casting. Therefore S is to be divided in proportion to the numbers 8, 7, 7, for A, B, C. Whence A's share is $\frac{8}{22} S = \frac{4}{11} S$.

P R O B. XV. (L. Diary, 1741.)

A person with six dice undertakes to bring up four faces of a sort at one throw, in seven trials. What is the odds against him?

The number of combinations of 4 aces out of 6 is $\frac{6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4} = 15$. And in any one case of these

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15, any two left out are capable of 25 variations, where no ace is found; therefore 25×15 , or 375, is the number of cases where only 4 aces can be cast. But because 5 or 6 aces may be cast, the number of chances for these must also be added.

Now the number of combinations of 5 aces out of 6, is 6; and since any one left out is capable of 5 variations where no ace is found; therefore 6×5 or 30 is the number of cases wherein only 5 aces can be cast with 6 dice. Lastly, there is but 1 case wherein 6 aces can be cast with 6 dice; therefore the sum of all the cases wherein 4 aces can be cast with 6 dice is $375 + 30 + 1 = 406$.

Now since there is the same variety for duces, trays, &c. therefore $406 \times 6 = 2436$ is the whole number of cases wherein 4 points of any sort can be cast. Put $s = 6^6 = 46656$, and $46656 - 2436 = 44220 = q$; then (by Cor. 1. Prop. IV.) the odds against the thrower will be as $\frac{q}{s}$ to 1 — $\frac{q}{s}$, that is, as .68705 to .31295, or as 2.1954 to 1.

PROB. XVI.

A undertakes with 5 dice, to throw 3 aces, a three, and a four at a throw; before B throws 2 duces, a five, and 2 fixes. To find their probabilities of winning.

All the chances on 5 dice amount to $6^5 = 7776$. Let a, b, c, d, e, f , stand for the ace, duce, 3, 4, 5, and 6. Then A's chance is a^3cd ; and B's chance is b^2ef^2 , where the indices only shew the repetitions of each letter. The number of permutations of 5 things a^3cd is $\frac{1.2.3.4.5}{1.2.3.1.1} = 20$ (by Prop. IV. Permutations.) And the number of permutations of the 5 things

5 things b^2ef^2 is $\frac{1.2.3.4.5}{1.2.1.1.2} = 30$. Therefore A's

chance of winning, is to B's, as $\frac{20}{7776}$ to $\frac{30}{7776}$, or as 2 to 3.

P R O B. XVII.

To find how many ways, three aces, two duces, a tray, and a four, can be cast with seven dice.

Let a, b, c, d, e, f , signify the ace, duce, tray, four, five and six, of any die. And since one face of every die is to appear, set down seven rows of them. Then a^3b^2cd will express the numbers.

Now the three a 's will fall in some of the seven rows, the two b 's in some of the remaining four, the c in one of the remaining two, and the d in the remaining one row, as they are set off in the figure.

The number of combinations of 3 a 's out of 7, is $\frac{7.6.5}{1.2.3} = 35$, and so often may 3 a 's be combined with the remaining 4 b 's.

The number of combinations of 2 b 's out of 4 is $\frac{4.3}{1.2} = 6$; therefore the number of combinations of the a 's and b 's is 35×6 . And so often may the a 's and b 's be combined with either of the two c 's.

The number of combinations of 1 c out of 2, is $\frac{2}{1} = 2$, therefore the number of combinations of the

the

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the *a*'s, *b*'s, and *c*'s, is $= 35 \times 6 \times 2$. And lastly, the combination of 1 *d* is but 1; therefore the number of all the combinations is $= 35 \times 6 \times 2 \times 1$, or $\frac{7.6.5}{1.2.3} \times \frac{4.3}{1.2} \times \frac{2}{1} \times \frac{1}{1} = \frac{1.2.3.4.5.6.7}{1.2.3 \times 1.2 \times 1 \times 1} = 420$, all the ways these can be cast, or the number of chances. But the number of all the chances on 7 dice is 6^7 , or 279936.

Cor. Hence the probability of casting 3 aces, 2 duces, 1 three, and 1 four, with 7 dice, at a throw is

$$\frac{420}{279936} = \frac{2}{1333} \text{ nearly.}$$

In general.

Let $a^p b^q c^r d^s$ &c. be proposed, where *p*, *q*, *r*, &c. denote the number of aces, duces, &c. or of any other faces that are named. Let $p+q+r+s$ &c. $= n$ = the number of dice.

Then observing the process that went before, we shall have $\frac{1.2.3.4 \text{ (to } n \text{ terms)}}{1.2.3(p \text{ terms}) \times 1.2.3(q) \times 1.2.3(r) \text{ \&c.}}$ = number of all the cases for casting *p* aces, *q* duces, &c.

P R O B. XVIII.

To find how many ways 3 faces of one sort, 2 of another sort, 1 of a third sort, and 1 of a fourth, can be cast with 7 dice.

Let $a^3 b^2 c^1 d^1$ denote the quantities, where *a*, *b*, *c*, *d*, stand for any four different faces; the indices 3, 2, 1, 1, the number of each.

Set down the 7 rows as in the last problem. Then it is plain, the combination of 3 *a*'s out of

D

7 =

$7 = \frac{7.6.5}{1.2.3}$; and that of 2 b 's out of 4 = $\frac{4.3}{2}$; that

of 1 c out of 2 = 2, therefore we have $\frac{7.6.5}{1.2.3} a^3 \times$

$\frac{4.3}{1.2} \times \frac{2}{1} c \times d$. But since a, b, c, d , may stand

for any sort, there will be 6 variations for a^3 (viz. $a^3, b^3, c^3, d^3, e^3, f^3$), and 5 variations for b^2 ; 4 variations for c , and 3 for d , and the combinations of c and $d = \frac{4.3}{2}$. Then the whole number of combi-

nations will be $\frac{7.6.5}{1.2.3} \times 6 \times \frac{4.3}{1.2} \times 5 \times \frac{2}{1} \times \frac{4.3}{2} =$

$\frac{7.6.5.4.3.2.1}{1.2.3 \times 1.2 \times 1} \times \frac{6.5.4.3}{1.2} = 420 \times 180 = 75600$.

Hence in general, let G = number of cases, when all the faces are named (as in Prob. XVII); l = number of quantities mentioned, and $H = \frac{6.5.4}{1.2.3}$ (l terms). Then GH = number of cases $R \times S \times T$ &c.

sought, where $R = 1.2$ when any index is twice repeated, or $1.2.3$, when thrice repeated, &c. and the like for S, T , &c.

EXAMPLE.

Suppose $a^3 b^2 c d$ as before, then $G = 420$, and $l = 4$, $R = 1$, $S = 1$, $T = 1.2$. Therefore GH

$= 420 \times \frac{6.5.4.3}{1.1.1.2} = 420 \times 180 = 75600$.

PROB. XIX.

If A and B be playing together, and A has got 1 of the game; what advantage has he of the game?

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If 2 be up, then A wants 1, and B wants 2; and therefore A's chance against B, is as 2 to 1.

If they be up in 1000, 10,000, &c. then A has no advantage over B, and their chances are equal, or as 1 to 1.

Therefore A's chance is always between 1 and 2, to B's 1. And the fewer games up, the nearer it approaches to 2 to 1, against B.

Otherwise, more exactly.

Let n be the number for being up, and A has got 1, put $\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}$ (to $n-1$ terms) $= x$; then A's advantage to B's, is as $1+x$ to $1-x$; and if they leave off play, the money must be divided in that proportion.

Exam. Suppose 4 be up, then $x = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} = \frac{5}{16}$, then A's share is $\frac{21}{16}$, and B's $\frac{11}{16}$, or rather $\frac{21}{32}$ and $\frac{11}{32}$.

P R O B. XX.

A and B playing together, A wants 2 of being up, and B wants 3; what is the advantage of either?

A's advantage or chance to B's chance, is as 3 to 2 nearly.

Or thus, more exactly.

It is evident the play will be ended in 4 games. Take all the unice of the 4th power of a binomial, $1 + 4 + 6 + 4 + 1$; take 2 of the first terms, $1 + 4 = 5$, for A; and 3 of the last terms for B $= 1 + 4 + 6 = 11$; and A's chance is to B's reciprocally as 11 to 5.

D 2

Or

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Or thus, more generally.

1. Let 1*l.* be the stake; and suppose each person wants 1 of being up, then the expectation of each (by Prop. V.) is $\frac{1}{2} \times 1*l.* = \frac{1}{2}$.

2. If A wants 1 of being up, and B 2. If A gets 1, he gets the stake 1; if B gets 1, then each wants 1 of being up, and they are in the same condition as in Art. 1. and A's expectation is $\frac{1}{2}$. Therefore (before the game be played) A has an equal chance to get the whole or the half, and therefore A's expectation = $\frac{1 + \frac{1}{2}}{2} = \frac{3}{4}$. And B's = $\frac{1}{4}$.

3. Let A want 1, and B 3. If A gets one he is up. If B gets 1 he will want 2 of being up; and they are in the same state as Art. 2. therefore A's expectation = $\frac{3}{4}$. Therefore (before the play) A has an equal chance for 1 or $\frac{3}{4}$, and his expectation = $\frac{1 + \frac{3}{4}}{2} = \frac{7}{8}$.

4. Let A want 1 and B 4; if A gets 1 he is up; if B gets 1, he wants 3, which is the case of Art. 3. and A's expectation is then = $\frac{7}{8}$, whence A has an equal chance for 1 or $\frac{7}{8}$; and his expectation = $\frac{1 + \frac{7}{8}}{2} = \frac{15}{16}$.

5. If A wants 1 and B 5; if A gets 1 he is up; if B gets 1, he wants 4, in which case (Art. 4.) A's expectation is $\frac{15}{16}$. Whence (before the play) he has an equal chance for 1 or $\frac{15}{16}$. Therefore his expectation = $\frac{1 + \frac{15}{16}}{2} = \frac{31}{32}$, and so forward.

6. Hence in general, if A wants 1, and B *n* of being up, then A's expectation = $\frac{2^n - 1}{2^n}$.

Again,

7. Suppose each wants 2 of being up; then the expectation of each is = $\frac{1}{2}$.

8. Sup-

8. Suppose A wants 2, and B 3; if A gets 1, then he will want 1 and B 3, then (Art. 3.) in that case, A's chance is worth $\frac{7}{8}$. But if B gets 1, they will both want 2, and then A's expectation = $\frac{1}{2}$. So that (before the play) A has an equal chance for $\frac{7}{8}$ or $\frac{1}{2}$. Whence A's expectation is = $\frac{\frac{7}{8} + \frac{1}{2}}{2} = \frac{11}{16}$.

9. If A wants 2 and B 4; if A gets the game, he wants 1 and B 4, as in Art. 4, where his exp. = $\frac{15}{16}$. But if B gets 1, then A wants 2 and B 3, as in Art. 8, where his exp. = $\frac{11}{16}$. Whence he has an equal chance for $\frac{15}{16}$ or $\frac{11}{16}$. And his expectation = $\frac{\frac{15}{16} + \frac{11}{16}}{2} = \frac{26}{32} = \frac{13}{16}$.

10. If A wants 2 and B 5; if A gets 1, he will want 1 and B 5; but if B gets 1, A will want 2 and B 4. Therefore (Art. 5, 9,) A will have an equal chance for $\frac{31}{32}$ or $\frac{26}{32}$, and his expectation = $\frac{\frac{31}{32} + \frac{26}{32}}{2} = \frac{57}{64}$.

11. If A wants 2 and B 6. If A gets 1, he will want 1 and B 6. But if B gets 1, A will want 2 and B 5. Whence (Art. 6, 10.) A has an equal chance for $\frac{63}{64}$ or $\frac{57}{64}$; and A's expectation = $\frac{120}{128}$. Again,

12. If A wants 3 and B 4. If A gets 1, then he wants 2 and B 4. But if B gets 1, then A wants 3 and B 3 (whose chances are equal = $\frac{1}{2}$). Therefore (Art. 11.) A has an equal chance for $\frac{13}{16}$ or $\frac{1}{2}$, and his expectation = $\frac{\frac{13}{16} + \frac{1}{2}}{2} = \frac{21}{32}$. And thus you may proceed as far as you will.

P R O B. XXI.

A and B play together; A wants 2 and B 3 of being up; and A's dexterity or skill is to B's, as 4 to 3; what is the odds of the game?

Since the play will be ended in 4 games; involve $a+b$ to the 4th power, $a^4 + 4a^3b + 6a^2bb + 4ab^3 + b^4$, where $a = 4$, $b = 3$. Then for A take all the terms where a is of 2 or more dimensions $a^4 + 4a^3b + 6a^2b^2 = 1888$; and for B, those where b is of 3 or more dimensions, $4ab^3 + b^4 = 513$; then A's chance is to B's, as 1888 to 513.

Otherwise.

Divide the skill, by the defect from being up; for the chances of each. Thus A's chance is to B's, as $\frac{4}{3}$ to $\frac{3}{2}$, or as 2 to 1 nearly; but not so exact as the former.

P R O B. XXII.

Three gamesters A, B, C, play together, whose proportion of skill is as a, b, c ; they play till A wants 1, B 2, C 3, of being up. What are their several chances?

Here the play will be concluded in 4 games; therefore raise the binomial $a + b + c$, to the 4th power, which will be $a^4 + 4a^3b + 6aabb + 12abbc + 4b^3c + 4bc^2 + 4ab^2c + b^4 + 4a^2c^2 + 12abcc + 6bbcc + 4ac^3 + c^4$.

Take all the terms for A, where a is of 1 or more dimensions, (where he wants 1) and where b and c are of fewer dimensions than 2 and 3 (the numbers they want). So will A's chance be $a^4 + 4a^3b + 12aabc + 4a^2c^2 + 12abcc + 6aacc$.

And B's $b^4 + 4b^3c + 6bbcc$.

And C's $4bc^3 + c^4$.

The

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The rest of the terms must be divided among them, according as they favour one or another of the players.

Otherwise.

Take the proportion of their skill, which divide respectively by what they want of being up. Thus the chances of A, B, C, will be $\frac{a}{1}, \frac{b}{2}, \frac{c}{3}$, nearly.

P R O B. XXIII.

A and B playing together, A has a chances for the game, and B has b chances; also A stakes p, and B stakes q; to find the advantage or disadvantage of the players.

Since A has a chances, and B has b, in any game; therefore in playing $a + b$ games, A will win a games, and B will win b. Therefore,

A gets $a \times p + q$, and wins $a \times p + q - a + b \times p = aq - bp$.

B gets $b \times p + q$, and wins $b \times p + q - a + b \times q = bp - aq$.

Therefore the advantage of A is $aq - bp$; and that of B, $bp - aq$. Therefore if $a : b :: p : q$, then $aq = bp$, and $aq - bp$, and also $bp - aq = 0$, and neither of them has the advantage. But if q is greater, then A has the advantage; but if p is greater, then B has the advantage.

P R O B. XXIV.

A and B playing together with n bowls a-piece; to find the probability of A's getting all his bowls at end.

Let a, b, c, d, e, f, &c. be all the bowls that both have, being equal to 2n in number; a, b, c, being

D 4

A's

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A's, and d, e, f , B's. The number of permutations that they are all capable of is $1.2.3.4$ (to $2n$ terms,) being the number of all the chances. The number of all the permutations, that the bowls d, e, f , (furthest from the jack) are capable of is $1.2.3$ (to n terms), when at the same time a, b, c , are nearest the jack. These are the number of chances for any one position of the bowls a, b, c , when d, e, f , are furthest from the jack. But since a, b, c , are likewise capable of the same number of permutations $1.2.3$ (to n terms), therefore $1.2.3$ (to n terms) $\times$ $1.2.3$ (to n terms) is the number of all the chances for a, b, c , being nearest, and d, e, f , furthest from the jack. Therefore $1.2.3$ (n terms) $\times$ $1.2.3$ (n terms) = the probability

$1.2.3.4$ (to $2n$ terms)
required; that is, $\frac{1.2.3 \text{ (to } n \text{ terms)}}{n+1.n+2.n+3 \text{ (to } n \text{ terms)}} =$
probability. Or thus, $\frac{n}{2n} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$ (to n
terms) = probability of all A's bowls being nearest
the jack.

P R O B. XXV.

A and B playing together with n bowls a-piece, to find the probability of A's getting m bowls at an end.

The number of permutations of $2n - m$ bowls, (the number furthest from the jack) is $1.2.3 \dots$ to $2n - m$. Here are n bowls of B, and $n - m$ bowls of A. But the number of combinations of $n - m$ things out of n , is $\frac{n.n-1}{1.2}$ (to $n - m$ terms); and so many different ways, will these permutations be varied, all which will become $\frac{n.n-1}{1.2}$ (to $n - m$ terms)

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terms) $\times 1.2.3 \dots$ to $2n - m$. Likewise the permutations of A's m bowls, will be $1.2.3 \dots$ to m ; and the number of permutations of $2n$ things being $1.2.3 \dots$ to $2n$, therefore the probability required is

$$\begin{aligned} & \frac{1.2.3 \text{ (to } m \text{ terms)} \times 1.2.3 \dots \text{ (to } 2n - m) \times \frac{n \cdot n - 1}{1.2} \text{ (} n - m \text{ terms)}}{1.2.3 \dots \text{ to } 2n} = \\ & \frac{\frac{1.2.3 \text{ (} m \text{)}}{1.2.3 \text{ (} n - m \text{ terms)}} \times \frac{1.2.3 \dots \text{ to } 2n}{2n \cdot 2n - 1 \text{ (} m \text{ terms)}} \times \frac{n \cdot n - 1 \text{ (} n - m \text{ terms)}}{1.2.3 \dots \text{ to } 2n}}{1.2.3 \text{ (} m \text{)}} \times \frac{n \cdot n - 1 \text{ (} n - m \text{ terms)}}{2n \cdot 2n - 1 \text{ (} m \text{ terms)}} = \\ & \frac{1.2.3 \text{ (} m \text{)}}{1.2.3 \text{ (} n - m \text{ terms)}} \times \frac{\frac{n \cdot n - 1 \dots \text{ to } 1}{1.2.3 \text{ (} m \text{)}}}{2n \cdot 2n - 1 \text{ (} m \text{ terms)}} = \\ & \frac{1.2.3 \text{ (} m \text{)}}{1.2.3 \text{ (} n - m \text{ terms)}} \times \frac{n \cdot n - 1 \dots \text{ to } 1}{2n \cdot 2n - 1 \text{ (} m \text{ terms)}} = \\ & \frac{1.2.3 \text{ (} n - m \text{ terms)} \times 2n \cdot 2n - 1 \text{ (} m \text{ terms)}}{n \cdot n - 1 \dots \text{ to } 1} = \\ & \frac{\frac{1.2.3 \dots \text{ to } n}{n \cdot n - 1 \text{ (} m \text{ terms)}} \times 2n \cdot 2n - 1 \text{ (} m \text{ terms)}}{n \cdot n - 1 \cdot n - 2 \text{ (to } m \text{ terms)}}, \text{ for the probability of getting } m \text{ bowls at an end.} \end{aligned}$$

P R O B: XXVI.

Any number (n) of things being given, a, b, c, d, &c. out of which m things are to be taken one by one. To find the probability of their being taken in the order they are written.

The number of permutations of n things taken m and m , is $n \times n - 1 \times n - 2$ &c. to m terms. But since the taking a, b, c , &c. in that order is but one case; therefore the probability of happening is

$$\frac{1}{n \cdot n - 1 \cdot n - 2 \text{ (to } m \text{ terms)}}$$

E X A M.

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E X A M:

What is the probability of taking 4 letters out of 6, in the order a, b, c, d.

Ans. $\frac{1}{6 \cdot 5 \cdot 4 \cdot 3} = \frac{1}{360} = \text{the probability.}$

P R O B. XXVII.

Any number (n) of things a, b, c, d, &c. being given; out of which m things are to be taken at once; to find the probability of taking those proposed.

The number of combinations of n things taken m at a time, is $\frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3}$ (to m terms); but the taking m things proposed is but one case of these, therefore $\frac{1 \cdot 2 \cdot 3}{n \cdot n-1 \cdot n-2}$ (to m terms) is the probability required.

E X A M.

What is the probability of taking the three d, e, f, out of the 6, a, b, c, d, e, f?

Ans. Here $n = 6$, $\frac{1 \cdot 2 \cdot 3}{6 \cdot 5 \cdot 4} = \frac{1}{20}$ is the probability.

P R O B. XXVIII.

What is the probability of throwing the six faces with 6 dice at a throw? And in how many throws may one undertake to do it?

Let a be the number of chances for happening, and b for failing. The number of permutations of 6 things, is the number of chances for happening, and that is $1 \times 2 \times 3 \times 4 \times 5 \times 6$. And the number

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number of all the chances is 6^6 ; therefore the probability of happening is $\frac{1.2.3.4.5.6}{6.6.6.6.6.6} = \frac{4.5}{6.6.6.6} = \frac{5}{324}$. Here $a = 5$, and $b = 324 - 5 = 319$. Therefore (by Cor. 2. Prob. XI.) $\frac{.7 \times 319}{5} = 44.8$; therefore in 44 or 45 throws, it is an equal chance for its happening or failing.

P R O B. XXIX.

Three persons A, B, C, draw blindfold 1 counter at a time, out of a heap containing 12 counters, 8 black and 4 white ones. They draw in the order A, B, C, A, B, &c. till one of them gets a white counter, and then wins. What is the probability of each for winning?

A's probability at first drawing is $\frac{4}{12}$, and his expectation of the stake $= \frac{4}{12} \times 1 = \frac{4}{12}$. This being taken out of the stake, there remains $1 - \frac{4}{12} = \frac{8}{12}$.

B's probability is $\frac{4}{11}$, and his expectation of the remaining stake is $\frac{8}{12} \times \frac{4}{11}$; then the remaining stake is $\frac{8}{12} - \frac{8.4}{12.11} = \frac{8.7}{12.11}$.

C's probability is $\frac{4}{10}$, and his expectation $= \frac{4.8.7}{12.11.10}$; and the remaining stake $= \frac{8.7}{12.11} - \frac{4.8.7}{10.12.11} = \frac{8.7.6}{12.11.10}$.

Then A's expectation again is $\frac{4.8.7.6}{12.11.10.9}$, and so on till the stake is gone.

Hence

Hence we have the series $\frac{4}{12}, \frac{4.8}{12.11}, \frac{4.8.7}{12.11.10},$
 $\frac{4.8.7.6}{12.11.10.9},$ &c. or $\frac{4}{12} + \frac{8}{11} P + \frac{7}{10} Q + \frac{6}{9} R$
 $+ \frac{5}{8} S$ &c. in which we must take 9 terms, in
 which a white counter must necessarily be drawn.
 Then take for A, the first, fourth, seventh terms;
 for B the second, fifth, eighth. For C the third,
 sixth, &c. And the sums will be the respective
 probabilities; that is, for A, $\frac{231}{495}$; for B, $\frac{159}{495}$;
 for C, $\frac{105}{495}$, which are as 77, 53, and 35.

And if there be more gamesters, the same series
 will answer all.

P R O B. XXX.

*Three gamesters A, B, C, throw in their turns a ball
 with 4 white faces and 8 black ones. What are the
 respective probabilities of throwing a white face first,
 and so to be the winner?*

Put $a = 4, b = 8, n = 12,$ Then A's probabi-
 lity at the first throw, is $\frac{a}{n}$ = his expectation; this
 taken out of the stake 1, is $1 - \frac{a}{n} = \frac{n-a}{n} = \frac{b}{n}.$

Then, as the number of faces do not diminish,
 B's probability will be also $\frac{a}{n},$ and his expectation
 $= \frac{ab}{nn};$ and the remaining stake $= \frac{b}{n} - \frac{ab}{nn} = \frac{bb}{nn}.$

In

In like manner C's probability is $\frac{a}{n}$ and his expectation $\frac{abb}{n^3}$; and so on.

Therefore we shall have the series $\frac{a}{n} + \frac{ab}{nn} + \frac{abb}{n^3} + \frac{ab^3}{n^4} + \frac{abb^4}{n^5}$, &c. Then take every third term (beginning at the first, second, third) for the respective expectations of A, B, C. But these 3 series, will be in geometrical progression, which therefore are to one another as $\frac{a}{n}, \frac{ab}{nn}, \frac{abb}{n^3}$; or as the numbers 9, 6, 4.

And if there be more gamesters, their expectations will be as $n, b, \frac{bb}{n}, \frac{b^3}{nn}$, &c. respectively.

If the rule of the play be, that each man throws twice together; then take the 2 first terms for A, the 2 next for B, and so on.

P R O B. XXXI.

There is a heap of counters, containing 4 white and 8 black ones. And A engages, in taking 7 out of the heap, that 3 shall be white ones. To find his chance.

If 12 counters in all give 4 white ones; then 7 counters will give $2\frac{1}{3}$ white ones. But as A was to take 3, his chance is $\frac{2\frac{1}{3}}{3}$ or $\frac{7}{9}$.

Otherwise.

All the cases in which 4 black ones out of 8 can be combined will be $\frac{8.7.6.5}{1.2.3.4} = 70$, and (in taking

out

out 7) 4×70 will be the number of cases, where in 4 black ones and 3 white ones (out of 4) can be combined = 280; because the combinations of 3 things out of 4 is 4.

But 4 white ones and 3 black ones may also happen; and the number of combinations of these is the same as the number of combinations of the three black ones alone, and that is $\frac{8.7.6}{1.2.3} = 56$; because the number of combinations of 4 white ones out of 4 is but 1; and $56 \times 1 = 56$.

Therefore the number of cases for happening is $280 + 56 = 336$. And the number of all the cases that 7 out of 12 can be combined is $\frac{12.11.10.9.8.7.6}{1.2.3.4.5.6.7} = 792$; therefore A's probability

of winning is $\frac{336}{792}$, and his chance of winning to the chance of losing, as 336 to 456, or as 14 to 19.

PROB. XXXII. (L. Diary, 1756.)

Suppose 12 half-pence thrown up, and those that come up heads to be taken away; and the remaining ones thrown up again, and so on; till at last all the half-pence are thrown up heads. In how many throws is it equally probable for happening or failing.

Since one time with another half will be taken away every throw; we must proceed to bisect the number till we come at 1, then the number of bisections + 1 = number of throws required. Thus 12, 6 (1), 3 (2), $1\frac{1}{2}$ (3), $\frac{3}{4}$ (4); whence between 4 and 5 throws are required.

Therefore take a series of terms in geometrical progression as 1 to 2, viz. 1, 2, 4, 8, 16, &c. and take that term which is nearest the number given, and

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and its place in the series will shew the number of throws.

P R O B. XXXIII.

In a lottery where there are 13333 blanks to a prize; what is the probability of getting a prize in 8000 tickets?

The probability is $\frac{8000}{13333} = \frac{8}{13}$ nearly.

P R O B. XXXIV.

In a lottery containing a great number of tickets, having 13333 blanks to a prize. How many tickets must be taken to make it an equal chance for getting a prize.

Here $a = 1$, $b = 13333$, and $t = 1$, then (by Prob. XV.) $t - .3 \times \frac{b}{a} + t - 1 = .7 \times \frac{b}{a} = .7 \times 13333 = 9333$ the number of tickets.

S C H O L.

It may be observed, that in many of these problems, to avoid more intricate methods of calculation, I have contented myself with a more lax method of calculating, by which I only approach near the truth. For some problems require such laborious computations, as no body can be able to go through; and which is worse, they often only serve for particular cases. And these I have endeavoured to avoid, upon account of the uncertainty of chance, as not requiring any such strictness. For in a single trial hardly any thing ever answers calculation; yet being often repeated it comes nearer and nearer.

Hence

Hence we may observe, that altho' we may by an equality of chance, contend, that a single event will happen in a certain number of trials; yet the same will not happen twice in double that number, nor thrice in tripple that number, &c. But the time will grow longer and longer; and therefore at long run, these events will not happen oftener than in the proportion of the chances they have for happening at first; but will keep constantly to that ratio.

Hence we may see, that in a great length of time, chance very little disturbs such events, as were designed to happen according to some determinate law; for however irregular these events may be at first, or in a few trials, yet in time they continually converge to a certain regular proportion; all irregularities continually correcting one another, so that these events continually tend to some determined rule. Whence follows this wonderful conclusion, that regularity arises out of irregularity, and order out of disorder.

And as we find that the happening of all events continually approach to some rule or law, according as experiments and observations increase. So on the other hand, when we find by multiplicity of observations, that these effects of chance ultimately converge to some determined law; we conclude that this is the very law, according to which these events were originally designed to happen.

And hence we may conclude, that all such laws, by which the various effects and productions of chance are regulated, are the effects of intelligence and design; and were instituted by some powerful agent, and intended to serve for some wise and useful purposes, which are necessary for preserving the order and œconomy of the universe. So that the great Author of Nature can make even Chance execute his designs.

A R T. II.

ANNUITIES UPON LIVES.

THIS is a branch of the Doctrine of Chances ; and besides, depends on the laws of mortality of mankind. The former has been briefly treated on ; and the latter depends entirely upon observation and experience. To proceed regularly in this affair, the rate of people dying every year must be extracted from the bills of mortality of some populous place or places, and put regularly into a table, which must stand as the basis of all calculations to be made for the purpose of annuities on lives.

N O T A T I O N.

J. AB, signifies the joint lives of the two persons A, B. And J. ABC, the joint lives of the three persons A, B, C.

L. AB, signifies the longest of the two lives A, B. And L. ABC, the longest of the three lives A, B, C, &c.

P R O P. I.

Supposing a number of children born in any year ; the decrements of life will be nearly in arithmetic progression ; or an equal number yearly will die every year, till they be all dead, which will be about 87 or 88 years after.

This appears from the following table, which is made by comparing four or five different tables with

E one

one another; which tables were each of them made from the bills of mortality of so many different places. So that this here being a mean among all these; is likely to be more true, and more generally useful than any one of the rest.

From this table it appears, that from the birth to the age of 7 or 8, mankind die faster; for out of 1000 that are born, 151 die in the first year, 108 in the second, 52 in the third, &c. But from the age of eight to twenty-two the fewest die, which is only 5 or 6 in a year; and afterwards 7, 8, or 9, till 80. And if 7 or 8 was to continue to die yearly, all the lives would then terminate about 88.

Now since these numbers, which are the differences of the people living each year, are so small in respect of the numbers themselves; and continue equal for large intervals of time; therefore the number of persons living year after year, for these intervals, will then be exactly in arithmetic progression; and through the whole from 8 to 88 may be accounted so without any sensible error.

A Table

*A Table of the people living at all ages ; from the Bills
of Mortality.*

Age. born.	livin. 1000	diff.	ag.	liv.	diff.	ag.	liv.	diff.	ag.	liv.	diff.
1	849	151	26	500	7	51	307	9	76	88	9
2	741	108	27	493	8	52	298	8	77	79	9
3	689	52	28	485	7	53	290	9	78	70	8
4	661	28	29	478	8	54	281	8	79	62	8
5	642	19	30	470	8	55	273	8	80	54	7
6	627	15	31	462	8	56	265	9	81	47	7
7	615	12	32	454	8	57	256	8	82	40	6
8	605	10	33	446	8	58	248	9	83	34	5
9	597	8	34	439	7	59	239	9	84	29	5
10	590	7	35	431	8	60	230	9	85	24	5
11	584	6	36	423	8	61	222	8	86	19	5
12	579	5	37	416	7	62	214	9	87	14	5
13	574	5	38	408	8	63	205	8	88	9	5
14	569	5	39	401	7	64	197	9	89	4	4
15	565	4	40	393	8	65	188	9	90	0	
16	560	5	41	385	8	66	180	8	31710 sum of all.		
17	555	5	42	377	7	67	171	9			
18	550	5	43	370	8	68	162	10			
19	545	5	44	362	8	69	152	9			
20	539	6	45	354	7	70	143	9			
21	533	6	46	347	8	71	134	10			
22	527	6	47	339	8	72	124	9			
23	521	7	48	331	8	73	115	9			
24	514	7	49	323	8	74	106	9			
25	507	7	50	315	8	75	97	9			

This table needs little explanation ; for the first col. is the age ; the second is the number of persons (out of 1000) living at that age ; the third shews their differences, or the persons that die in a year, at that age.

Cor. If a right line be drawn, and divided into 88 equal parts, and at the several points of division, per-

pendiculars be erected proportional to the numbers in the table, and a line be drawn through the tops thereof; this figure will represent the state of mankind in all ages; and will be very nearly a triangle.

If there be taken 78 children, each 10 years old; then one will die every year, till they be all extinct.

PROP. II. PROB.

To shew the use of the foregoing table, in matters relating to human life.

1. *To find how many people there are of any age, compared with the whole number of people in a nation. Suppose it be required to know how many persons there are from 21 to 25 years old (inclusive). Add up these 5 numbers, from 21 to 25, and the sum is 2602. And the whole number by the table is*

31710. Therefore $\frac{2602}{31710} \times$ number of inhabi-

tants, shews how many there are of these ages.

2. *To find how many years, it is an even wager, whether a person of a given age shall live or die; as suppose his age be 38. Take the number against 38, which is 408, and half it, which will be 204; then seek it in the table, which will be at 63. Therefore it is an even wager that he lives till 63, that is 25 years.*

3. *To find the difference of insurance upon different lives; as between 30 and 50. Here it is 470 to 8, or 59 to 1, that a man of 30 dies not in a year; and only 315 to 8, or 39 to 1, for a man of 50. The price of insurance then ought to be as 59 to 39, or as 3 to 2 nearly.*

4. *To find the probability of a person of a given age a , living any number of years t . Let N = number of persons living at the age a . L = persons living*

living at the age $a + t$, then $\frac{L}{N}$ = probability of his living t years. And $\frac{N - L}{N}$ = probability of his being dead in t years.

5. To find the chances of two lives, A and B, whose ages are a and b . Thus by the table, let t = any time, and

N = number of persons living at the age a .

n = persons living at the age b .

L = persons living at the age $a + t$.

l = persons living at the age $b + t$.

$D = N - L$ = persons dead in t years (after a).

$d = n - l$ = persons dead in t years (after b).

Then,

$\frac{Ll}{Nn}$ = probability of both A and B living t years.

$\frac{Dd}{Nn}$ = probability of both being dead in t years.

$1 - \frac{Dd}{Nn}$, or $\frac{Nn - Dd}{Nn}$ = probability of one of them living t years.

$\frac{Ld}{Nn}$ = probability of A being alive and B dead in t years.

6. To find the chances for three lives, A, B, C, whose ages are a , b , c .

Denoting the numbers as in Art. 5. And moreover putting p = persons living at the age c , m = persons living at the age $c + t$, $f = p - m$ = persons dead in t years after c .

Then,

$\frac{Llm}{Nnp}$ = probability of all three living t years.

$\frac{Ddf}{Nnp}$ = probability of all being dead in t years.

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$1 - \frac{Ddf}{Nnp}$, or $\frac{Nnp - Ddf}{Nnp}$ = probability of some of them living t years.

$\frac{Ldf}{Nnp}$ = probability of A being alive and the rest dead in t years.

$\frac{Llf}{Nnp}$ = probability of A and B being alive, and C dead in t years.

7. To find the value of a sum of money, due t years hence, for one or more lives. It is plain the purchaser ought only to pay such a part of the present value, as there are chances for the persons living. Therefore (by Art. 4, 5, or 6,) find the probability of the person or persons living t years hence, and multiply this by the present worth of 1*l.* and that product by the sum given, for the value required. As for example, to find the value of 100*l.* due 12 years hence; upon the longest of two lives, aged 40 and 50, at 4 *p.* cent. Here (Art. 5.) $t = 12$. $N = 393$, $n = 315$, $D = 393 - 298 = 95$, $d = 315 - 214 = 101$. And present value of 1*l.* = .6246; and the probability = $\frac{Nn - Dd}{Nn} =$

$\frac{114200}{123705} = .9225$. Then $.6246 \times .9225 \times 100 = 57^{11} \frac{62}{100}$, the present value.

8. To find the value of an annuity for one or more lives, or to continue t years, if the persons live as long. By Art. 7. find the present value of the annual rent for 1 year, for 2 years, for 3 years, and so on till you get t years; then the sum of all these is the value required for t years. But if t is not given, this process is to be continued, till the terms be 0, or so small as to be inconsiderable, which will be at the extremity of old age. This often requires a la.

a laborious calculation, and therefore shorter methods will be shewn after.

9. *Any number of equal ages being given; to find the value of the longest life.* Take the number in the table against the common age given, and divide it by the number of lives. Seek the quotient (or the next less) in the table, and take the age against it, from which subtract the given age; the remainder shews how long an annuity certain is to continue, which is equal in value to the longest life.

Suppose 5 persons, of 38 years old. Against 38 is 408, and $\frac{408}{5} = 81\frac{1}{2}$, and against 81 is 77; whence $77 - 38 = 39$ years continuance for the value of the longest life.

10. *To find the value of the reversion, after one or more lives.* Find the value of the annuity for the life or lives proposed, by Art. 8; and subtract it from the value of the perpetuity.

11. *To find the complement of life for any age by this table, or any such table made from the bills of mortality.*

Find the difference between the number of that age, and the number of 5 years difference. Then as that difference, to 5 :: so is the number of either age, to its complement. Or instead of 5 years, one may take any other difference of years.

Exam. Take the ages 5 and 10; then $642 - 590 = 52$, and $52 : 5 :: 590 : 57$, the comp. life for 10. This makes the extremity of old age only 67, so that people die very fast at this age.

Again, for the ages 20 and 25, $539 - 507 = 32$, and $32 : 5 :: 539 : 84$ the comp. of life for 20; which makes the extremity of old age 104, so that now people die very slow.

Lastly, take the ages 48 and 53, $331 - 290 = 41$, and $41 : 5 :: 290 : 35$, the comp. life for 53,

and $35 + 53 = 88$, the extremity of old age at this rate.

If this be done for every 5 or every 10 ages from 8 to 88, by this table; the mean quantity for the extremity of old age, will come out 94.

12. *Any number of equal lives being given; to find the time in which any number of them will fail.*

Let p be the number of all the lives, and q the number of these to die. Take the number against the given age in the table; and take $\frac{p-q}{p} \times$ that number, and seek it in the table; against which is the age that q of them will be dead. The difference of these ages shews the time.

Exam. Suppose 12 lives of 40, and 5 to fail. Against 40 is 393, and $\frac{7}{12} \times 393 = 229$, against which is 60 nearly; then $60 - 40 = 20$ years, the time required.

13. *To find the probability of a person of a given age (a) living t years; and afterwards dying in v years.*

Find the number p in the table against his age a , also the number q against $a + t$, and the number r against $a + t + v$; then the probability required is $= \frac{q-r}{p}$.

Suppose the age $a = 48$, $t = 13$, $v = 5$; then $p = 331$, $q = 222$, $r = 180$, then the probability

$$= \frac{222 - 180}{331} = \frac{42}{331}.$$

P R O P. III. PROB.

To find the expectation of a single life, or the time a person of a given age, may reasonably expect to live.

This

This has been resolved by the table in Art. 2. of the last Prop. I shall here resolve it by calculation, supposing the decrements of life to be equal. Let x be the time sought, a the persons age, and $88 - a = n =$ the complement of life; then the probability of living x years is $= \frac{x}{n}$, and supposing it an equal chance for living or dying in that time, we have $\frac{x}{n} = \frac{1}{2}$, whence $x = \frac{1}{2} n$, or half the complement of life.

Exam. Suppose $a = 38$, then $n = 50$, and $\frac{1}{2} n = 25$, his expectation of life.

PROP. IV. PROB.

The ages of two persons being given, to find the expectation of their joint lives; or how long they may reasonably expect to live together.

Let n, m , be the complements of life, n being the least. Then the probability of their joint continuance for 1, 2, 3, &c. years, will be $\frac{m-1}{m} \times \frac{n-1}{n} + \frac{m-2}{m} \times \frac{n-2}{n} + \frac{m-3}{m} \times \frac{n-3}{n}$ &c. to n terms; that is,

$$\frac{1}{mn} \times \text{into } mn - 1 \times \overline{m + n + 1}$$

$$\begin{aligned} &+ mn - 2 \times \overline{m + n} + 4 \\ &+ mn - 3 \times \overline{m + n} + 9 \\ &+ mn - 4 \times \overline{m + n} + 16 \\ &\&c. \text{ to } n \text{ terms.} \end{aligned}$$

But

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But the sum of all the first column $= n \times mn$,

And the sum of the second column $= -\frac{n+1}{2}$
 $n \times \overline{m+n}$

And the sum of the third column $= n \times \frac{n+1}{2} \times$
 $\frac{2n+1}{3}$, by the differential method.

And the whole sum $= \frac{mn^2}{2} - \frac{mn}{2} - \frac{n^3-n}{6}$; and
 dividing by mn , the expectation will be $=$
 $\frac{n-1}{2} - \frac{nn-1}{6m}$.

But this may otherwise be solved by finding n
 terms of the series $\frac{mn}{mn} + \frac{m-1}{m} \times \frac{n-1}{n} + \frac{m-2}{m}$
 $\times \frac{n-2}{n}$ &c. that is,

$$\begin{aligned} \frac{1}{mn} \times \text{into} \quad & mn - 0 \times \overline{m+n} + 0 \\ & + mn - 1 \times \overline{m+n} + 1 \\ & + mn - 2 \times \overline{m+n} + 4 \\ & + mn - 3 \times \overline{m+n} + 9 \\ & \text{\&c.} \end{aligned}$$

And the sum found as the foregoing, and divid-
 ed by mn , will be $\frac{1}{2}n - \frac{nn}{6m} + \frac{1}{2} + \frac{1}{6m}$, that
 is, $\frac{n+1}{2} - \frac{nn-1}{6m}$, for the expectation. And
 taking the mean between this and the former, we
 shall have $\frac{n}{2} - \frac{nn-1}{6m}$, for the true expectation.

And neglecting $\frac{1}{6m}$ as being very small, the value
 of

of the expectation of the two joint lives will be

$$\frac{n}{2} - \frac{nn}{6m}.$$

Cor. If both the lives are equal, the expectation will be $\frac{1}{2}n$, or $\frac{1}{2}$ their common complement.

PROP. V. PROB.

The ages of three persons being given, to find the expectation of their joint lives; or the time they may expect to live together.

Let n, m, p , be their complements of life, and n the least. Then the probabilities of continuing together for 1, 2, &c. years will be $\frac{m-1}{m} \times$

$$\frac{n-1}{n} \times \frac{p-1}{p} + \frac{m-2}{m} \times \frac{n-2}{n} \times \frac{p-2}{p} +$$

&c. to n terms; that is, $\frac{1}{mnp} \times$ into

$$\begin{aligned} & mnp - 1 \times R + 1 \times S - 1 \\ & + mnp - 2 \times R + 4 S - 8 \\ & + mnp - 3 R + 9 S - 27 \\ & + mnp - 4 R + 16 S - 64 \end{aligned}$$

&c. where $R = mn + mp + np$. $S = m + n + p$.

Then by the differential method,

The sum of the first column $= n \times mnp$.

The second column $= -\frac{n+1}{2} \times n R$.

The third column $= n \cdot \frac{n+1}{2} \cdot \frac{2n+1}{3} S$.

The fourth column $= -\frac{n^4 + 2n^3 + n^2}{4}$.

And the sum of all, divided by mnp , will be

$$\frac{n-1}{2} - \frac{n^2-1}{6p} - \frac{n^2-1}{6m} + \frac{n^2-1}{12mp} n.$$

But

But if it be solved by finding n terms of the series,

$$\begin{aligned} mnp &= 0 \\ + mnp &= 1 R + 1 S = 1. \\ + mnp &= 2 R + 4 S = 8. \\ + mnp &= 3 R + 9 S = 27. \\ &\&c. \end{aligned}$$

Then the result will be $\frac{n+1}{2} - \frac{n^2-1}{6p} - \frac{n^3-1}{6m}$
 $+ \frac{n^3-1}{12mp} n.$

And the mean between the two is $\frac{n}{2} - \frac{n^2-1}{6p}$
 $- \frac{n^3-1}{6m} + \frac{n^3-1}{12mp} n$, for the expectation of 3 joint
 lives. Or leaving out some very small quantities,
 the expectation will be $\frac{n}{2} - \frac{n^2}{6m} - \frac{n^2}{6p} + \frac{n^3}{12mp}.$

Cor. If all the lives are equal, then the expectation
 will be $\frac{1}{4} n.$

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After the same manner the expectation of 4 joint
 lives would be $\frac{1}{2} n - \frac{nn}{6m} - \frac{nn}{6p} - \frac{nn}{6q} + \frac{n^3}{12mp}$
 $+ \frac{n^3}{12mq} + \frac{n^3}{12pq} - \frac{n^4}{20mpq}$, where q is the com-
 plement of the 4th life.

And if all the lives are equal, the expectation
 $= \frac{1}{4} n.$ And in general, if r be the number of
 equal lives, then the expectation of the joint lives
 will be $\frac{n}{r+1}.$

PROP. VI. PROB.

To find the expectation of the longest of two or more
 lives whose ages are given.

Let

Let $m, n, p, q, \&c.$ be the complements of the several ages from the youngest to the oldest; that is, m being the greatest, n the next, and so on in order. Then $\frac{1}{2}m + \frac{nn}{2.3m} + \frac{p^2}{3.4mn} + \frac{q^3}{4.5mnp} + \&c.$ (continued to as many terms as there are persons) will be the expectation of the longest life.

For in one life the expectation will be $\frac{1}{2}m$; and in two lives the second person has a chance to outlive the first, which must be added to the former, and this chance is $\frac{nn}{6m}$; being the same chance as the value of the shortest life was diminished, for two joint lives (Prop. IV.)

And in three lives, the third has a chance to outlive the other two, which is $\frac{p^2}{12mn}$, this must be added to the former.

And in four lives, the fourth has a chance to outlive the other three, which is also to be added, and is $\frac{q^3}{20mnp}$; and so on.

Cor. 1. *If all the lives are equal, and r their number; then $\frac{rn}{r+1} = \text{expectation of the longest liver.}$*

For in two lives the expectation $= \frac{1}{2}n + \frac{1}{6}n = \frac{4}{6}n = \frac{2}{3}n$.

And in three lives, it is $= \frac{1}{2}n + \frac{1}{6}n + \frac{1}{12}n = \frac{9}{12}n = \frac{3}{4}n$.

And in four lives it is $= \frac{1}{2}n + \frac{1}{6}n + \frac{1}{12}n + \frac{1}{20}n = \frac{3}{4}n + \frac{1}{20}n = \frac{16}{20}n = \frac{4}{5}n$, and so on.

Cor.

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Cor. 2. *If there be r equal lives; the time wherein g of them will fail will be $\frac{g}{r+1} n$.*

For if $g = r$, then the time $= \frac{r}{r+1} n$, by Cor. 1,
and if $g = 1$, the time $= \frac{1}{r+1} n$, by Schol.
Prop. V. and there is the same reason for the rest.

PROP. VII.

The joint lives of two persons + the longest life of the same two persons is = sum of their single lives.

Let the two persons be A, B. Their joint lives is the same as the shortest life; and which ever dies first (as A), being the shorter life; the other (B that remains) is the longer life; so that one always makes up the other; and their sum is the life of A + the life of B. Therefore the joint lives of A, B + longest life of A, B = life of A + life of B.

Cor. *The longest of two lives is equal to the sum of the single lives — the joint lives.*

PROP. VIII.

If ABCFGH be several people, and Z a life equal to the longest life of ABC; and Y a life equal to the longest of the rest FGH. Then the value of the longest life of ZY is = value of the longest life of the whole number ABCFGH.

For the longest life of ABCFGH is the very same as the longest life of the two parts ABC and FGH, or of the two Z and Y. Consequently the present

present values are equal; that is, value of $l.$ ABCFGH = value of $l.$ ZY. Where $l.$ signifies longest life.

Cor. 1. If $Z = \text{longest life of } AB$; then longest life of $ZF = \text{longest life of } ABF$. Or $l. ZF = l. ABF$.

Cor. 2. If $Z = l. ABC$; then $l. ZF = l. ABCF$.

Cor. 3. If $Z = l. AB$; $Y = l. FG$; then $l. ZY = l. ABFG$, &c.

PROP. IX.

If ABCFGH &c. be several lives, and Z a life equal to the joint lives of ABC ; and Y a life equal to the joint lives of the rest FGH . Then the value of the joint lives of $ZY = \text{value of the joint lives of all } ABCFGH$. Or $j. ZY = j. ABCFGH$.

For the joint lives of any number of people, is the shortest life among them. And the shortest life of ABCFGH is the same thing as the shortest life of the two parcels ABC and FGH , or of the two lives Z and Y ; that is, the joint lives of ABCFGH = joint lives of ZY , and therefore their present values are equal.

Cor. 1. If $Z = \text{joint lives of } AB$; then the joint lives of $ZF = \text{joint lives of } ABF$; or $j. ZF = j. ABF$.

Cor. 2. If $Z = \text{joint lives of } ABC$; then $j. ZF = j. ABCF$.

Cor. 3. If $Z = \text{joint lives of } AB$, $Y = \text{joint lives of } FG$, then $j. ZY = j. ABFG$.

And so of others.

Cor. 4. If A, B, C , be three lives, and $Z = l. AB$; then $j. \overline{CZ} = j. AC + j. BC - j. ABC$.

For

For j . $\overline{CZ} = j$. $\overline{C}$ and l . $AB = j$. $\overline{C}$ and $A + B - j$. AB (Prop. VII.) $= j$. $AC + j$. $BC - j$. ABC .

Cor. 5. If A, B, C , be three lives, and $Z = j$. AB ; then l . $\overline{CZ} = l$. $AC + l$. $BC - l$. ABC .

For l . $\overline{CZ} = l$. $\overline{C}$ and j . $AB = l$. $\overline{C}$ and $A + B - l$. AB (Prop. VII.) $= l$. $AC + l$. $BC - l$. ABC . And so of others.

PROP. X.

The value of an annuity for the longest life of any number of persons ABCFGH — the value of the longest life of any part of them ABC = value of the reversion of the longest life of the rest FGH after the longest life of ABC. Or l . ABCFGH — l . ABC = rever. l . FGH after l . ABC, where l . signifies the longest life.

For the interest or advantage that ABC have in the annuity, and also what FGH have in it, is equal to the whole advantage that they all (ABCFGH) have in it, and that is for the longest life of ABCFGH. Therefore

From the value of the longest life of ABCFGH, Subtract the value of the longest life of ABC, The remainder is the value of the reversion of the longest of the lives of FGH after those of ABC. That is, l . ABCFGH — l . ABC = rever. l . FGH after l . ABC.

Cor. 1. The value of l . AF — life of A = rever. F after A.

Cor. 2. Value of l . AFG — a = rever. l . FG after A, where a = life of A.

Cor. 3. Value of l . ABF — l . AB = rever. F after l . AB.

&c.

PROP.

PROP. XI.

The value of an annuity for the joint lives of FGH — the value of the joint lives of ABCFGH = value of the reversion of the joint lives of FGH after the joint lives of the rest ABC. Or $j. FGH - j. ABCFGH = \text{rev. } j. FGH \text{ after } j. ABC.$

Put $Y = j. ABC$, $Z = j. FGH$; then $j. YZ = j. ABCFGH$. But $j. YZ =$ is the same as the shorter life, which here is Y because Z ($j. FGH$) is supposed the value of the whole. Therefore $j. ZY$ being the life of Y , $Z - j. ZY =$ life of Z after Y ; that is, (restoring the letters) $j. FGH - j. ABCFGH = \text{rev. } j. FGH \text{ after } j. ABC.$

Cor. 1. *Joint lives of FG — joint lives of AFG = rev. $j. FG$ after A.*

Cor. 2. *$j. FGH - j. AFGH = \text{rev. } j. FGH$ after A.*

Cor. 3. *$F - j. ABF = \text{rev. } F$ after $j. AB$.*

Cor. 4. *$F - j. ABCF = \text{rev. } F$ after $j. ABC$.*

Cor. 5. *$j. FG - j. ABFG = \text{rev. } j. FG$ after $j. AB$.*

SCHOL.

By help of the foregoing propositions, the value of annuities for joint lives, or for longer lives, when they are many, or for reversions of such lives, may be far easier found, than by the laborious and intricate methods hitherto used for these purposes. And indeed many problems cannot be solved by that way, which are easily managed by this new and short method.

F

PROP.

PROP. XII. PROB.

*Supposing the decrements of life to be equal; to find the present value of 1*l.* due *t* years hence, if a person of a given age lives so long.*

Let *n* be the complement of the person's age; then since the money is to be valued according to the chance there is for the person's living, and that chance is $\frac{n-t}{n}$; therefore $\frac{n-t}{n}$ is to be multiplied into the present worth of 1*l.* due *t* years hence, which is $\frac{1}{R^t}$ where *R* is the amount of 1*l.* for a year.

Therefore the present value will be $\frac{n-t}{n} \times \frac{1}{R^t}$
or $\frac{n-t}{nR^t}$.

Cor. 1. If *S* is any sum of money due *t* years hence for a life; its value = $\frac{n-t}{nR^t} S$.

Cor. 2. If several sums of money *S*, *T*, *U*, be due on a life, *t*, *v*, *w*, years hence. The present value = $\frac{n-t}{nR^t} S + \frac{n-v}{nR^v} T + \frac{n-w}{nR^w} U$.

EXAM.

What is the present value of 1*l.* due 10 years hence, if a person 40 years old lives so long; at 4 per cent.

By Tab. I. $\frac{1}{R^t} = .6755$, $n = 88 - 40 = 48$;

and *t* = 10, and *n* - *t* = 38, then $\frac{38}{88} \times .6755 =$
l. *s.* *d.*
292 = 5 - 10.

PROP.

PROP. XIII. PROB.

To find the present value of an annuity of 1*l.* a year, for *t* years; if a person of a given age lives so long. But if he does not, the annuity is to cease at his death.

Let *n* be the complement of the given age; and *R* = amount of 1*l.* for a year, *r* = its interest. Suppose the several payments not to be at 1, 2, 3, &c. years distance, but sooner by the time *v*; then 1 - *v*, 2 - *v*, 3 - *v*, &c. will be the times of the payments. Therefore (by Cor. 2. Prop. XII.) the present value of the annuity will be $\frac{n+v-1}{nR^{1-v}}$

+ $\frac{n+v-2}{nR^{2-v}}$ + $\frac{n+v-3}{nR^{3-v}}$ &c. to *t* terms; for in *t* years the annuity is supposed to be ended. And this will be divided into the two series,

$$R^v \times \frac{n+v}{n} \times \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} \text{ \&c. to } t \text{ terms.}$$

$$- \frac{R^v}{n} \times \frac{1}{R} + \frac{2}{R^2} + \frac{3}{R^3} \text{ \&c. to } t \text{ terms.}$$

The former series = $\frac{n+v}{n} R^v \times \frac{1 - \frac{1}{R^t}}{r}$ (Prop. XXVI. proportion) and the latter = $-\frac{R^v}{n} \times :$

$$\frac{R}{rr} - \frac{t}{rR^t} - \frac{R}{rrR^t} : (\text{increments, Prop. XXX.}) \text{ put}$$

$$p = \frac{1}{R^t}, \text{ and the sum of the two series} = R^v$$

$$\times \frac{n+v}{n} \times \frac{1-p}{r} - \frac{R}{nrr} + \frac{tp}{nr} + \frac{Rp}{nrr} : =$$

$$\frac{n + v - \overline{n + v. p + tp}}{nr} R^v - \frac{1 - p}{nrr} R^{1+v} = va.$$

lue of the annuity, where $p = \frac{1}{R}$.

Cor. 1. *If the payments begin at a year's end, $v=0$, and then the value* $= \frac{1 - p.n + pt}{nr} - \frac{1 - p}{nrr} R.$

Cor. 2. *If the payments begin immediately, add 1 to the value found by Cor. 1.*

EXAMPLE.

Suppose the age be 56, and the annuity be for 16 years, at 5 per cent. Here (by Tab. I.) $p = .458$, and $1 - p = .542$, $n = 32$, $t = 16$. Then by Cor. 1.

$$\frac{1 - p.n + pt}{nr} = 15.42$$

$$\frac{1 - p}{nrr} R = 7.11$$

$$\underline{8.31} \quad \text{the value req.}$$

If the payments begin immediately, the value $= 9.31$.

PROP. XIV. PROB.

To find the present value of an annuity of 1l. for the life of a given age.

1 Way.

I shall suppose, as in the last Prop. that the first payment is not exactly at a year's distance, but sooner by the time v ; and that the decrements of life are equal, and that the extremity of old age is

88, as agreeable to my general table. Now the several rents being so many sums of money payable at different times, all at a year's distance; therefore proceeding as in the last Prop. (according to Cor. 2. Prop. XII.) the present value of the annuity will be

$$\frac{n+v-1}{n R^{1-v}} + \frac{n+v-2}{n R^{2-v}} + \frac{n+v-3}{n R^{3-v}} \text{ \&c. to } n \text{ terms; for in } n \text{ years the life}$$

will be extinct. This consisting of two series, the sum will be found, as in the last Prop. to be $R^v \times$ into $\frac{n+v}{n} \times \frac{1-p}{r} - \frac{R}{nrr} + \frac{np}{nr} + \frac{Rp}{nrr}$: that is,

$$\text{the value of the annuity} = \frac{n+v-vp}{nr} R^v -$$

$$\frac{1-p}{nrr} R^{1+v}; \text{ where } p = \frac{1}{R^n} \text{ the present value of 1l. annuity due } n \text{ years hence.}$$

Cor. 1. If P is the present worth of 1l. annuity to continue n years certain; then the value of the annuity for the life whose complement is n , is =

$$\frac{n-R-vr.P}{nr} R^v; \text{ or for } R^v \text{ take } \frac{1}{1+rv}.$$

$$\text{For (by Prob. XXXIV, Algebra) } P = \frac{R^n-1}{rR^n}$$

$$\text{whence } R^n = \frac{1}{1-rP}, \text{ and } \frac{1}{R^n} = 1-rP = p,$$

which put for p , will give the rule as above.

Cor. 2. If the payments begin at the year's end;

$$v=0, \text{ and the value} = \frac{1}{r} - \frac{1-p}{nrr} R = \frac{n-RP}{nr}.$$

Cor. 3. If the payments begin immediately, then

$$v=1, \text{ and the value} = \frac{nr-1+p}{nrr} R = \frac{n-P}{nr} R.$$

Or else add 1 to the value, by Cor. 2.

ANNUITIES.

EXAMPLE.

Suppose a life of 40, interest 4 per cent. $n = 48$,
 $R = 1.04$, $r = .04$; $p = .152$. $P = 21.195$.
 Then by Cor. 2.

$$\frac{1}{r} = 25.00,$$

Or,

$$\frac{1-p}{nr} R = 11.49.$$

$$\frac{n-RP}{nr} = 13.52$$

$$\text{Annuity} = \underline{13.51}$$

Or if the annuity commence at present $13.52 + 1 = 14.52$, the value. Otherwise by Cor. 3.

$$\frac{nr - 1 + p}{nr} R = \frac{1.115}{.0768} = 14.52$$

$$\text{Or } \frac{n-P}{nr} R = \frac{27.877}{1.92} = 14.52$$

2. Otherwise thus.

Let the complement of life be n , R the amount of 1*l.* for a year, r its interest, then the probability of living 1, 2, 3, &c. years is $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c. and the present value of all the rents

will be $\frac{n-1}{nR} + \frac{n-2}{nR^2} + \frac{n-3}{nR^3}$ &c. for n terms;

but as $R = 1 + r$, $R^2 = 1 + 2r$, $R^3 = 1 + 3r$, &c. very near, we shall have the present value aforesaid, that is, the value of the annuity =

$$\frac{n-1}{n.1+r} + \frac{n-2}{n.1+2r} + \frac{n-3}{n.1+3r}, \text{ \&c. to } n \text{ terms.}$$

And to find the sum S by the method of increments,

ments, here s , or the $z+1^{\text{th}}$ term $= \frac{n-z+1}{n \times 1+z+1.r}$;

put $v = 1 + z + 1.r$, then $z+1 = \frac{v-1}{r}$, and

$$= \frac{n - \frac{v-1}{r}}{nv} = \frac{nr + 1 - v}{nr v} = \frac{nr + 1}{nr} \times \frac{1}{v} -$$

$\frac{v}{nr v}$. But by Ex. 12, Prop. XIII. Increments, the

$$\text{Integral of } \frac{1}{v} = \frac{b. \log. v}{v} - \frac{1}{2v} - \frac{v}{12vv} \&c.$$

But $v = rz = r$, because $z = 1$; therefore

$$\text{Int. of } \frac{1}{v} = \frac{b. \log. v}{r} - \frac{1}{2v} - \frac{r}{12vv} \&c. \text{ Also}$$

(by form 1, or Cor. Ex. 10,) the integ. of $\frac{v}{nr v}$ or

$$\frac{1}{nr} = \frac{v}{nr v} = \frac{v}{nr r}. \text{ Therefore int. } \frac{nr + 1}{nr v} - \frac{1}{nr}$$

$$= \frac{nr + 1}{nr} \times \text{into } \frac{b. \log. v}{r} - \frac{1}{2v} - \frac{r}{12vv} : -$$

$$\frac{v}{nr r}. \text{ But when } s = 0, z = 0, \text{ and } v = 1 + r =$$

$$R, \text{ therefore } s \text{ or } 0 = \frac{nr + 1}{nr} \times \frac{b. \log. R}{r} - \frac{1}{2R}$$

$$- \frac{r}{12vv} : - \frac{R}{nr r}, \text{ which subtracted gives } s =$$

$$\frac{nr + 1}{nr} \times \frac{b. \log. v}{r} - \frac{1}{2v} - \frac{r}{12vv} : - \frac{v}{nr r} +$$

$$\frac{nr + 1}{nr} \times - \frac{b. \log. R}{r} + \frac{1}{2R} + \frac{r}{12RR} : +$$

$$\frac{R}{nr r}. \text{ But } \frac{v-R}{nr r} = \frac{zr}{nr r} = \frac{z}{nr}. \text{ Therefore put-}$$

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$$\text{ting } Q = \frac{nr + 1}{nr} \times \frac{b. \log. R}{r} - \frac{1}{2R} - \frac{r}{12RR} + \frac{z}{nr}; \text{ the value of the annuity} = \frac{nr + 1}{nr} \times \text{into} \\ \frac{b. \log. v}{r} - \frac{1}{2v} - \frac{r}{12vv} : - Q.$$

E X A M.

Suppose a life of 40 as before, $n = 48 = z$,
 $R = 1.04$, $r = .04$, $v = 2.96$, then $\frac{nr + 1}{nr} =$
 $\frac{2.92}{1.92} = 1.5208$, and $\frac{z}{nr} = \frac{1}{r} = 25$; whence Q
 $= 25.755$, and
 $\frac{nr + 1}{nr} \times \frac{b. \log. v}{r} - \frac{1}{2v} - \frac{r}{12vv} : - Q =$
 $40.996 - 25.755 = 15.24$, for the value of the
 annuity; which is something too great.

3. Or thus.

Let n be the complement of life, then sum up
 $n - 1$ terms in Table II, the sum of which divide
 by n ; and the quotient will be the value of that
 life whose complement is n .

This is so easy it needs no example.

To prove the rule; suppose there are n persons,
 and that one dies every year, and each of them dies
 before the year be out. Then the first gets no-
 thing; but the rest get the value of an annuity
 certain for 1, 2, 3, &c. years, to $n - 1$ years in-
 clusive. The sum of all which is $n - 1$ terms of
 Table II. And since the number of men is n , there-
 fore that sum is to be divided by n , for the mean
 value of a single life whose complement is n . And
 this exactly agrees with Cor. 2. before going.

S C H O L.

SCHOL.

This Prob. may be solved near enough, without introducing R^v into the computation, by finding the sum of $\frac{n-1}{nR} + \frac{n-2}{nR^2} + \frac{n-3}{nR^3}$ &c. to n terms, and multiplying the last result by R^v or by $1 + v$.

From this Prop. and its Cors. tables may be constructed, to shew the present worth of an annuity of 1*l.* for the life of any age; and such a table you have farther on, for several rates of interest. And as the easiest way of constructing such tables, is by using p or P , that is, the present worth of 1*l.* or of the annuity certain of 1*l.*; therefore I have also inserted two tables for that purpose at the same rates of interest.

I must take notice that this Prob. may be solved, though not so exactly, by the *expectation* of life, thus. Find the value of the annuity certain, for the time $\frac{1}{2}n$, which in this example will be 15.25; but it will always give something too much. The reason is, the former methods find the present worth of the annuity every year, and so gets the mean value. But the method by the expectation of life, only takes the present worth of the mean age, which is a different thing. But as the difference is not very great, it may serve in several cases for an approximation.

If we suppose money to bear no interest; then in the preceding calculations, $R = 1$; and then we have nothing to do, but to sum up an arithmetical progression, which is $\frac{1}{2}n$, or half the complement of life; which shews that when money bears no interest, the value of a life coincides with the expectation of life, being equal to an annuity for $\frac{1}{2}n$ years, without interest. But this method applied to the foregoing example, would make the value

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value of such a life 24*l.* which is far from the truth, being almost double. Therefore in using the expectation of life, we must allow interest.

PROP. XV. PROB.

*The ages of two persons being given ; to find the value of an annuity of 1*l.* for their joint lives ; that is, till one of them dies.*

I Way.

Let the complements of life be m and n , n being the least. R = amount of 1*l.* for a year, r its interest. Then proceeding as before, the probability of each living a year being $\frac{m-1}{m}$, $\frac{n-1}{n}$; therefore the probability of both living 1, 2, 3, &c. years, will be $\frac{m-1}{m} \times \frac{n-1}{n}$, $\frac{m-2}{m} \times \frac{n-2}{n}$, $\frac{m-3}{m} \times \frac{n-3}{n}$ &c. whence the value of the joint lives will be $\frac{m-1}{m} \times \frac{n-1}{nR} + \frac{m-2}{m} \times \frac{n-2}{nR^2} + \frac{m-3}{m} \times \frac{n-3}{nR^3}$ &c. to n terms; for in n years one must necessarily die. That is,

$$\begin{aligned} \frac{1}{mn} \times \text{into} \quad & \frac{mn}{R} - 1. \frac{m+n}{R} + \frac{1}{R} \\ & + \frac{mn}{R^2} - 2. \frac{m+n}{R^2} + \frac{4}{R^2} \\ & + \frac{mn}{R^3} - 3. \frac{m+n}{R^3} + \frac{9}{R^3} \\ & \text{\&c.} \end{aligned}$$

$$\begin{aligned} \text{But the sum of the first column} &= mn \times \frac{1 - \frac{1}{R^n}}{r} \\ &= mn \times \frac{1 - p}{r} \quad (\text{by Prop. XXVI. proportion}) \text{ and} \\ &\quad \text{the} \end{aligned}$$

the second column $= \frac{m+n}{r} \times \frac{pn}{r} + \frac{pR}{rr} - \frac{m+n}{rr}$

R (by Increments, Prop. XXX.)

And the third col. $= \frac{RR+R}{r^3} - p \times \frac{nn}{r} + \frac{2nR}{rr} + \frac{RR+R}{r^3}$: (ib.) and the whole divided by

mn . Now the sum of all these is $\frac{1}{mn} \times$ into $\frac{mn}{r}$

$$- \frac{m+n}{rr} R + \frac{R+1}{r^3} R + \frac{m-n}{rr} p R - \frac{R+1}{r^3}$$

$p R$; where $p = \frac{1}{R^n}$ the present worth of 1*l.* due n years hence. Therefore the value of the annuity

$$= \frac{1}{mn} \times \text{into } \frac{mn}{r} - \frac{m+n-m-n.p}{rr} R + \frac{R+1}{r^3}$$

$\times R \times 1-p$. If the payments begin sooner than after a year, suppose sooner by the time v ; then multiply the result by R^v , or by $1+vr$. This is the most exact way of any.

Cor. 1. If the lives are equal, the value of the joint lives will be $= \frac{nr-2R}{nrr} + \frac{RR+R}{nrr^3} \times 1-p$, exactly.

2. Otherwise thus.

Let M, N , be the values of the two single lives, and put $q = \frac{1}{R^m}$, $s = 1-p$, $t = 1-q$,

$f = p+1$. And assume $\frac{x}{M+1 \times N+1 - R \times MN}$
 $=$ value of the joint lives; to find x .

By

By the last Prop. $M = \frac{1}{r} \times 1 - \frac{sR}{nr}$, and N

$$= \frac{1}{r} \times 1 - \frac{tR}{mr}, \text{ and } M + 1 \times N + 1 = \frac{RR}{rr} \times$$

$$1 - \frac{s}{nr} \times 1 - \frac{t}{mr} = \frac{RR}{rr} \times 1 - \frac{ms + nt}{mnr} + \frac{st}{mnrr}$$

$$\text{And } RMN = \frac{R}{rr} \times 1 - \frac{ms + nt}{mnr} R + \frac{stRR}{mnrr}$$

$$\text{therefore } M + 1 \times N + 1 - RMN = \frac{RR - R}{rr}$$

$$+ \frac{RR - R^3}{mnrr^3} st = \frac{Rr}{rr} - \frac{rstRR}{mnrr^3} = \frac{R}{r} - \frac{stRR}{mnrr^3}$$

$$\text{Therefore } \frac{x}{M + 1 \times N + 1 - RMN} =$$

$$\frac{\frac{R}{r} - \frac{stRR}{mnrr^3}}{\frac{R}{r} - \frac{stRR}{mnrr^3}} = \text{value of the annuity found the first}$$

$$\text{way} = \frac{1}{r} - \frac{ms + nc}{mnrr} R + \frac{RR + R}{mnrr^3} s = A.$$

$$\text{And } x = \frac{R}{rr} \times 1 - \frac{ms + nc}{mnr} R + \frac{RR + R}{mnrr} s -$$

$$\frac{stR}{mnrr} A, \text{ from which subtract}$$

$$MNR = \frac{R}{rr} \times 1 - \frac{ms + nt}{mnr} R + \frac{stRR}{mnrr}, \text{ and}$$

$$\text{we have } x - MNR = \frac{R}{rr} \times - \frac{nc - nt}{mnr} R +$$

$$\frac{RR + R - stRR}{mnrr} s - \frac{stR}{mnrr} A, \text{ therefore } MNR$$

$$\text{falls short of the true value } x, \text{ by } \frac{RR}{rr} \times \text{ into } -$$

$$\frac{p + q}{mr} + \frac{R + 1 - tR}{mnrr} s - \frac{st}{mnrr} A = y, \text{ whence}$$

$$MNR$$

$\frac{MNR + y}{M + 1 \times N + 1 - MNR} =$ value of the two joint lives. But since y is always very small and affirmative, it may be set aside, and then the value

of the joint lives will be $\frac{MNR}{M + 1 \times N + 1 - MNR}$,

being a small matter too little, and is one of Mr. Moivre's rules.

It may here be observed, that since the correction y is irregular, the rule is not genuine; that is, not derived from true principles.

Cor. 2. *The value of the two joint lives is =*
 $\frac{MNR}{M + N - rMN}$, *nearly.*

For $\frac{MNR}{M + 1 \times N + 1 - R MN} =$
 $\frac{MNR}{MN + M + N + 1 - MN - rMN} =$

$\frac{MNR}{M + N + 1 - rMN} = \frac{MNR}{M + N - rMN}$;
 for throwing out the 1 in the denominator, helps to correct it and bring it near the truth. Or as Mr.

Moivre has it, $\frac{MN}{M + N - rMN}$, leaving out R .
 A very easy rule, but is something too little.

Cor. 3. *If the lives are equal, then the value of the joint lives will be =* $\frac{M}{2 - rM}$, *or rather* $\frac{RM}{2 - rM}$.

E X A M.

Let the ages be 40 and 50, interest at 5 per cent.
 Then $m = 48$, $n = 38$, $p = .1566$; and $\frac{1}{r} =$
 $m +$

$$\frac{m+n-m-n.p}{mnrr} R = 36480 - \frac{90.3 - 1.644}{.0025}$$

$$= 1017.72, \text{ and } \frac{RR+R}{r^3} \times 1-p = 14523, \text{ and}$$

$$\frac{1017.72 + 14523}{mn = 1824} = 8.25, \text{ for the value of the joint lives.}$$

Or thus.

By Tab. III. $M=1209$, and $N=10.68$. Whence $MN = 129.12$, and $M+N = 22.77$. And $MNR = 135.57$, and $MNr = 6.45$; therefore

$$\frac{135.57}{22.77 - 6.45} = \frac{135.57}{16.32} = 8.30, \text{ the value.}$$

3. Otherwise.

The same may be also resolved by the method of increments, another way, as in the last Prop.

The value of the joint lives was $\frac{m-1}{m} \times \frac{n-1}{nR}$

$$+ \frac{m-2}{m} \times \frac{n-2}{nR^2} + \frac{m-3}{m} \times \frac{n-3}{nR^3} \&c. =$$

$$\frac{m-1}{m} \times \frac{n-1}{n.1+r} + \frac{m-2}{m} \times \frac{n-2}{n.1+2r} \&c.$$

Then to find n terms or $z-1$ terms of this series,

$$\text{we shall have the } z^{\text{th}} \text{ term or } s = \frac{m-z \times n-z}{mn \times 1+ zr},$$

$$\text{put } 1+ zr = v, \text{ then } z = \frac{v-1}{r}, \text{ whence } s =$$

$$\frac{rm+1-v \times rn+1-v}{rrmnv} = \frac{p-v \times q-v}{rrmnv}, \text{ (by}$$

$$\text{substitution) } = \frac{pq-p+q.v+vv}{rrmnv} = \text{(putting } f$$

=

$= p + q) \frac{1}{rrmn} \times \text{into } v - f + \frac{pq}{v}$, and the in-

$$\text{tegral} = \frac{1}{rrmn} \times \text{into } \frac{v^v}{2r} - \frac{fv}{r} + pq \times$$

$$\frac{b. \log. v}{r} - \frac{1}{2v} - \frac{r}{12vv}, \text{ because } v = r. \text{ Or } s =$$

$$\frac{1}{rrmn} \times \text{into } \frac{v - r - 2f}{2r} v + pq \times \frac{b. l. v}{r} - \frac{1}{2v}$$

&c. but when s is $= 0$, $z - 1 = 0$, and $z = 1$,
whence $v = 1 + r = R$; therefore the int. correc-

$$\text{ted is } s = \frac{1}{mnrr} \times \text{into } \frac{v - r - 2f. v + 2f - 1. R}{2r}$$

$$+ pq \times : \frac{b. l. v - b. l. R}{r} + \frac{1}{2R} - \frac{1}{2v} :$$

Thus in the former example, $m = 48$, $n = 38$,
 $r = .05$; $z - 1 = 38$, or $z = 39$, and $v = 2.95$,
 $rm = 2.4$, $rn = 1.9$, $f = 6.3$, $pq = 9.86$, and $rrmn$
 $= 4.56$, whence

$$\frac{v - r - 2f. v + 2f - 1. R}{2r} = -164.35$$

$$\text{and } pq \times \&c. = 206.73$$

$$\underline{42.38}$$

And $\frac{42.38}{4.56} = 9.10$ the value of the annuity, which
is something too great.

PROP. XVI. PROB.

*The ages of three persons being given, to find the
value of an annuity of 1*l.* for their joint lives.*

1 Way.

Let the complements of their lives l, m, n ; n
being the least. R the amount of 1*l.* for a year,
 r its

its interest. Then by reasoning as in the last Prop. The value of the joint lives will be $\frac{l-1}{l}$

$\times \frac{m-1}{m} \times \frac{n-1}{nR} + \frac{l-2}{l} \times \frac{m-2}{m} \times \frac{n-2}{nR^2} + \frac{l-3}{l} \times \frac{m-3}{m} \times \frac{n-3}{nR^3}$ &c. to n terms: that is, (putting $S = lm + ln + mn$, $T = l + m + n$) the value $= \frac{1}{lmn} \times$ into

$$\begin{aligned} & \frac{lmn}{R} - \frac{1S}{R} + \frac{1T}{R} - \frac{1}{R} \\ & + \frac{lmn}{R^2} - \frac{2S}{R^2} + \frac{4T}{R^2} - \frac{8}{R^2} \\ & + \frac{lmn}{R^3} - \frac{3S}{R^3} + \frac{9T}{R^3} - \frac{27}{R^3} \\ & + \frac{lmn}{R^4} - \frac{4S}{R^4} + \frac{16T}{R^4} - \frac{64}{R^4} \\ & \text{\&c.} \end{aligned}$$

Then finding the sum of each column (by the method of increments, Prob. XXX.) as in the last Prop. we shall have

$$1 \text{ col.} = lmn \times \frac{1-p}{r}, \text{ where } p = \frac{1}{R^n}.$$

$$2 \text{ col.} = Sp \times \frac{n}{r} + \frac{R}{rr} - \frac{SR}{rr}.$$

$$3 \text{ col.} = T \times : \frac{R^2 + R}{r^3} - p \times : \frac{nn}{r} + \frac{2nR}{rr} + \frac{R^2 + R}{r^3}.$$

$$4 \text{ col.} = - \frac{R^3 + 4R^2 + R}{r^4} + p \times : \frac{n^3}{r} + \frac{3nnR}{rr} + \frac{3nR.R + 1}{r^3} + \frac{R^3 + 4R^2 + R}{r^4} : \text{ and the whole sum} =$$

$$\begin{aligned}
 &= \frac{lmn}{r} + \frac{3n-2T}{rr} \times pnR - \frac{SR}{rr} \times \overline{1-p} + \\
 &\frac{R^2+R}{r^3} \times \overline{T-Tp+3np} - \frac{R^3+4R^2+R}{r^4} \times \\
 &\overline{1-p} = \frac{lmn}{r} + \frac{3n-2T \times pnR - S \times \overline{1-p} R}{rr} \\
 &+ \frac{R^2+R}{r^3} \times \overline{T-Tp+3np} - \frac{R^3+4R^2+R}{r^4} \times \\
 &\overline{1-p}. \text{ And dividing by } lmn, \text{ the value of the} \\
 &\text{joint lives will be} = \frac{1}{lmn} \times \text{into } \frac{lmn}{r} + \\
 &\frac{n-2T \times pnR - RS \times \overline{1-p}}{rr} + \frac{RR+R}{r^3} \times \\
 &\overline{T-Tp+3np} - \frac{RR+4R+1}{r^4} \times R \times \overline{1-p}.
 \end{aligned}$$

Cor. 1. If the lives are all equal; the value of the annuity upon three joint lives will be $\frac{1}{n^3} \times \text{into} : \frac{n^3}{r} -$
 $\frac{nnR}{rr} + \frac{3n \times RR + R}{r^3} - \frac{R^2 + 4R + 1}{r^4} R \times$
 $\overline{1-p}.$

2. Otherwise thus.

Let M, N, Q, be the values of the single lives, and Z a life equal to the joint lives M, N; then by the last Prop. Cor. 2. we shall find $\bar{Z} = \frac{MN}{M+N-rMN}$, and by the same, the value of

the joint lives Z and Q, $\frac{MN}{M+N-rMN} \times Q,$
G will

will be $\frac{ZQ}{Z+Q-rZQ} = \frac{MN}{M+N-rMN} + Q-rQ$

$$\frac{MN}{M+N-rMN} = (\text{multiplying by } M+N-rMN)$$

$$\frac{MNQ}{MN + Q \times M + N - rMN - rQ \times MN} =$$

$\frac{MNQ}{MN + MQ + NQ - 2rMNQ}$; which is a rule of Mr. Moivre's, but is something too small, partly by leaving out R.

Cor. 2. If all the lives are equal, the value of the three joint lives will be $\frac{M}{3-2rM}$.

E X A M. I.

Suppose the three ages 45, 56, and 68; whose complements are $l = 43$, $m = 32$, $n = 20$. Interest at 4 per cent.

Here $lmn = 27520$, $p = .4564$, $1-p = .5436$, $R = 1.04$, $r = .04$, $S = 2876$, $T = 95$. Then

$$\begin{aligned} \frac{lmn}{r} &= \frac{27520}{.04} = & + 688000 \\ + \frac{3n-2T \times pnR - SR \times 1-p}{rr} &= - 1787000 \\ + \frac{RR+R}{r^2} \times \frac{T-Tp+3np}{1-p} &= + 2619700 \\ - \frac{R^2+4R+1}{r^4} \times 1-p.R &= - 1378500 \\ &+ 3307700 \\ &- 3165500 \\ &142200 \end{aligned}$$

Then $\frac{142200}{27520} = 5.16$ the value of the annuity required.

3. Or

3. Or thus.

By Tab. III. $M=12.68$, $N=10.47$, $Q=7.33$.

Then

$$\begin{array}{rcl} MNQ & = & 973.05, \quad MN = 132.74 \\ 2rMNQ & = & 77.84, \quad MQ = 92.94 \\ & & NQ = 76.74 \end{array}$$

$$\begin{array}{rcl} MN + MQ + NQ & = & 302.43 \\ - 2rMNQ & = & - 77.84 \\ \hline & & 224.59 \end{array}$$

Then $\frac{973.05}{224.59} = 4.33$ for the value of the three joint lives, which is too little; but if it be multiplied by $RR = 1.08$, the product is 4.67 more exact.

EXAMPLE II.

Suppose each life be 68; then $n = 20$, and $n^2 = 400$. $R = 1.04$, $r = .04$. Then

$$\begin{array}{rcl} \frac{n^3}{r} & = & + 200000 \\ - \frac{3nnR}{rr} & = & - 780000 \\ + \frac{3n \times R^2 + R}{r^3} & = & + 1989000 \\ - \frac{R^3 + 4R + 1}{r^4} R \times \frac{1}{1-p} & = & - 1378500 \\ & & + 2189000 \\ & & - 2158500 \\ & & + 30500 \end{array}$$

Then $\frac{30500}{8000} = 3.81$, the value of the joint lives.

G 2

Or

Or thus.

$$\begin{array}{r}
 \text{By Table III. } M = 7.33; \quad 7.33 \\
 \quad \quad \quad .08 \\
 \hline
 2r M = .5864 \\
 \quad \quad \quad 3.0000 \\
 \hline
 \quad \quad \quad 2.4136 \\
 \hline
 \end{array}$$

Then $\frac{7.33}{2.41} = 3.04$ the value of the joint lives, too little.

SCHOL.

Mr. Moivre has given another general rule by the expectation of life, which is this. Add 1 to each age, and then find the expectation of these joint lives, by Prop. 3d, 4th, or 5th. Find the complement of twice this number to 88, and take it as a single life, which being found in Tab. III. will give the value required. At the same time he informs us, that he does not pretend that any of these rules are strictly true, but only near approximations.

PROP. XVII. PROB.

To find the value of an annuity of 1l. for four, five, six, or any number of joint lives, of any ages.

1 Way.

This Prob. is to be resolved by finding successively the value of three lives, and then the value of four, and then of five, and so on, by the foregoing propositions. Thus, let A, B, C, D, E, &c. be any number of persons, then

1. To find the value of the four lives ABCD; first find the value of the joint lives of the three ABC,

ABC, by the last Prop. and let Z be a single life equal to these three. Then by Prop. XV. find the value of the joint lives of ZD, which will be the value of the four joint lives ABCD, by Prop. IX.

2. To find the value of the lives ABCDE. Having found the value of the four joint lives ABCD as before, put a single life Z = these four; then by Prop. XV. find the value of the joint lives ZE, and that will be the value of the joint lives ABCDE, by Prop. IX.

3. Again, put Z = joint lives of ABCDE, and then find the joint lives of ZF, which will be the joint lives of ABCDEF. And thus you may go on as far as you will.

E X A M.

To find the value of the four joint lives whose ages are 31, 45, 56 and 68, at 4 per cent. The value of the three joint lives, whose ages are 45, 56 and 68, is $5.18 = Z$, by the last Prop. Then by Table III. the value of a life of 31, is 14.81; then (by Cor. 2. Prop. XV.) $5.16 \times 14.81 = 76.42$, and $.04 \times 76.42 = 3.06$; and $5.16 + 14.81 - 3.06 = 16.91$; and $\frac{76.42}{16.91} = 4.52$, for the value of the four joint lives.

2. Or thus.

Seek the value of the life 5.16 ($= Z$) in Tab. III. against which, or the nearest to it, is the age 75, whose complement is 13. Also the complement of the given age (31) is 57. Then by the two complements 13 (n), and 57 (m), the value of the joint lives 31 and 75 is easily found by Prop. XV. first way.

S C H O L.

To avoid a deal of trouble in finding the value of many joint lives, take the following approximation,

G 3

tion,

tion. Add all the ages together, which divide by the number of lives, the quotient is the mean age. By the help of this, find the value of the annuity for so many equal joint lives, by the following proposition.

PROP. XVIII. PROB.

To find the value of an annuity of 1*l.* for several equal joint lives, whose number is *m*, and complement of life *n*.

RULE.

The present value = $A - \frac{B}{n} + \frac{C}{nn} - \frac{D}{n^3} + \frac{E}{n^4} - \frac{F}{n^5} + \&c. + \frac{Z}{n^m} \times \frac{1}{1-p}$, continued to *m* + 1 terms. Where $p = \frac{1}{R^n}$, the present worth of 1*l.* due *n* years hence, and *Z* the last term, which is always to be $\times$ d by $1-p$. And the co-efficients are,

$$rA = 1, \text{ or } A = \frac{1}{r}.$$

$$rB = m \times \frac{1}{A + 1}, \text{ or } B = \frac{mR}{rr}.$$

$$rC = m \cdot \frac{m-1}{2} \times \frac{R}{r} + \frac{m-1}{m-1} \cdot B,$$

$$rD = m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \times \frac{R}{r} + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot B + \frac{m-2}{m-2} \cdot C.$$

$$rE = m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \times \frac{R}{r} + \frac{m-1}{1} \cdot \frac{m-2}{2} \cdot \frac{m-3}{3} \cdot B + \frac{m-2}{1} \cdot \frac{m-3}{2} \cdot C.$$

$$\frac{m-2}{2} \cdot \frac{m-3}{3} B + \frac{m-2}{1} \cdot \frac{m-3}{2} C +$$

$\frac{m-3}{1} D$. And so on, where the progression is evident.

This is proved by induction as follows; we have by the corollaries of Prop. 14, 15, 16, the value of

$$1 \text{ life} = \frac{1}{r} - \frac{R}{nrr} \times \overline{1-p}.$$

$$2 \text{ lives} = \frac{1}{r} - \frac{2R}{nrr} + \frac{RR+R}{n^2r^2} \times \overline{1-p}.$$

$$3 \text{ lives} = \frac{1}{r} - \frac{3R}{nrr} + \frac{3 \times \overline{RR+R}}{n^2r^2} -$$

$$\frac{R^2 + 4R^2 + R}{n^3r^3} \times \overline{1-p}.$$

&c. where the law of progression is evident. And the calculation of the quantities A, B, C, &c. is founded upon Prob. 30th of my *Increments*. For

it will be found that $\frac{R}{rr}, \frac{RR+R}{r^2}, \frac{R^2+4RR+R}{r^3},$

&c. (the terms into which $1-p$ is multiplied) are the last terms in the series for summing up the laterals, squares, cubes, &c. as in Prob. 30, of the *Increments*. But it is not necessary to stay to reduce the coefficients to these forms, but take them as they lie in the rule, when you make use of it.

PROP. XIX. PROB.

To find the value of an annuity of 1l. upon the longest of two lives, whose ages are given.

1 Way.

Let m and n be the complements of their ages to 88. The probability of each living one year

G 4

will

will be $\frac{m-1}{m}$ and $\frac{n-1}{n}$, and the probability of each failing will be $1 - \frac{m-1}{m}$, $1 - \frac{n-1}{n}$, or $\frac{1}{m}$ and $\frac{1}{n}$; and the probability of both failing in a year will be $\frac{1}{m} \times \frac{1}{n}$ or $\frac{1}{mn}$, and therefore $1 - \frac{1}{mn}$ is the probability of some being alive at the year's end.

Again, the probability of both failing the second year will be $\frac{2}{m} \times \frac{2}{n}$ or $\frac{4}{mn}$, and $1 - \frac{4}{mn}$ is the probability of some being alive at two year's end.

After the same manner the probability of some being living after 3, 4, &c. years will be $1 - \frac{9}{mn}$, $1 - \frac{16}{mn}$ &c.

Therefore the present values being found for 1, 2, 3, &c. years, the value of the annuity will be $\frac{1}{R} - \frac{1}{mnR} + \frac{1}{R^2} - \frac{4}{mnR^2} + \frac{1}{R^3} - \frac{9}{mnR^3} + \frac{1}{R^4} - \frac{16}{mnR^4}$ &c. to n terms at least, in which time one must necessarily die. But since the other may live longer, the series must be continued to more terms, put $v = \frac{m+n}{2}$, and the series must be continued to v terms nearly. But this consists of two series

$$\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} \text{ \&c. for } v \text{ terms, } = 1 - \frac{1}{R^v} - \frac{1}{mn}$$

$$\frac{1}{mn} \times \frac{1}{R} + \frac{4}{R^2} + \frac{9}{R^3} \&c. (v \text{ terms}) = \frac{p}{mn} \times$$

$$\frac{vv}{r} + \frac{2Rv}{rr} - \frac{1-p}{mn} \times \frac{R^2+R}{r^3}, \text{ (Increments,$$

Prob. 30.) putting $p = \frac{1}{R^v}$, to be found in Tab. I.

$$\text{Therefore the value of the annuity} = \frac{1-p}{r} +$$

$$\frac{p}{mn} \times \frac{vv}{r} + \frac{2Rv}{rr} - \frac{1-p}{mn} \times \frac{R+1}{r^3} R, \text{ a near approximation.}$$

If the lives are equal, $m = n = v$.

2. Or thus.

Put M , N , for the values of the single lives, j . $\overline{MN}$ for their joint lives; then by Cor. 1. Prop. VII. the value of the longest of the two lives is $= M + N - j. \overline{MN}$.

If the lives are equal, then the value of the longest life will be $2M - \frac{MR}{2 - rM}$.

EXAMPLE.

Let the ages be 45 and 56, then $m = 43$, $n = 32$, $v = 37\frac{1}{2}$. $R = 1.04$. $p = .230$, $1-p = .770$.

$$\text{Then } \frac{1-p}{r} = 19.25$$

$$\frac{p}{mn} \times \frac{vv}{r} + \frac{2Rv}{rr} = 14.02$$

$$33.27$$

$$- \frac{1-p}{mn} \times \frac{R+1}{r^3} R = - 18.55$$

$$\text{The value required } 14.72$$

Or

ANNUITIES.

Or thus.

The value of the joint lives found by Prop. XV,
is 8.28; then by Tab. III. $M = 12.68$

$$N = 10.47$$

$$23.15$$

$$- 8.28$$

$$\text{The value required} \quad 14.87$$

PROP. XX. PROB.

To find the value of an annuity of 1l. upon the longest of three lives, whose ages are given.

1 Way.

Let l, m, n , be their complements to 88; then $\frac{l-1}{l}, \frac{m-1}{m}, \frac{n-1}{n}$ are the probabilities of living one year, and $1 - \frac{l-1}{l}, 1 - \frac{m-1}{m}, 1 - \frac{n-1}{n}$, or $\frac{1}{l}, \frac{1}{m}, \frac{1}{n}$, are the probabilities of failing in a year, and $\frac{1}{lmn} =$ probability of all failing; and $1 - \frac{1}{lmn} =$ probability of some living a year,

Likewise $\frac{2}{l}, \frac{2}{m}, \frac{2}{n}$ are the probabilities of failing the second year, and $\frac{2}{l} \times \frac{2}{m} \times \frac{2}{n}$ or $\frac{8}{lmn} =$ probability of all failing the second year; and $1 - \frac{8}{lmn} =$ probability of some living two years.

Also

Also $1 - \frac{27}{lmn}$, $1 - \frac{64}{lmn}$ &c. = probabilities of some living 3, 4, &c. years. Then finding the present worth for 1, 2, 3, &c. years, and putting $v = \frac{l+m+n}{3}$; the value of the annuity will be $\frac{1}{R} +$

$\frac{1}{R^2} + \frac{1}{R^3}$ &c. (for v terms) $= \frac{1-p}{r}$, where $p = \frac{1}{R^v}$.

$-\frac{1}{lmn} \times$ into $\frac{1}{R} + \frac{8}{R^2} + \frac{27}{R^3}$ &c. (v terms) $=$

(Prob. 30, Increments) $-\frac{1}{lmn} \times$ into $\frac{R^3+4R^2+R}{r^4}$

$-p \times : \frac{v^3}{r} + \frac{3vvR}{rr} + \frac{3vR \times \overline{R+1}}{r^3} +$

$\frac{R^3+4R^2+R}{r^4}$: therefore the whole sum, or va-

lue of the annuity is $= \frac{1-p}{r} + \frac{p}{lmn} \times : \frac{v^3}{r} +$

$\frac{3vvR}{rr} + \frac{3vR}{r^3} \times \overline{R+1} + \frac{R^3+4R^2+R}{r^4}$:

$-\frac{1}{lmn} \times \frac{R^3+4R^2+R}{r^4}$.

2. Or thus.

This problem may also be solved pretty near the truth, by the expectation of life, thus;

Let l, m, n , be the complements of life as before, and x = years certain that the annuity is to continue. Then the probability that some one will be alive, and the rest dead, at the end of x years, is

$\frac{l-x}{lmn} xx + \frac{m-x}{lmn} xx + \frac{n-x}{lmn} xx$; and the pro-

bability

bability that they shall all be dead in x years,

$\frac{x^3}{lmn}$, which is equal to the other *per* quest. Therefore

$l - x + m - x + n - x = x$, and $4x = l + m + n$,

and $x = \frac{l + m + n}{4}$. A near approximation, but

something too little.

By the same reasoning if there were 4 lives, then

$x = \frac{k + l + m + n}{5}$; and universally, if N be the

number of lives, then $x = \frac{k + l + m + n \&c}{N + 1}$; which

will come the nearer to truth, as the number N is greater.

3. Otherwise thus.

By Prop. XIX. find a life Z equal to the longest of two of them A, B ; and then (by the same Prop.) find the value of the longest of the two Z and C (a third life); and that will be the value of the annuity, by Cor. 1. Prop. VIII.

4. Or thus.

$A + B + C - j. AB - j. ZC = l. ABC$. That is, take the sum of the three single lives, from which subtract the sum of the joint lives of AB , and of ZC , for the value of the longest of three lives.

For $l. AB = A + B - j. AB$, and $l. ABC$ or $l. ZC = Z + C - j. ZC = l. AB + C - j. ZC = A + B - j. AB + C - j. ZC$.

5. Or thus.

$A + B + C - j. AB - j. AC - j. BC + j. ABC$. That is, to the sum of the three single lives add the value of the three joint lives, from which

which subtract the sum of the joint lives taken two and two. Which is Mr. Moivre's rule.

For by Cor. 4, Prop. IX. $j. ZC = j. AC + j. BC - j. ABC$, whence $A + B + C - j. AB - j. ZC = A + B + C - j. AB - j. AC - j. BC + j. ABC$.

E X A M P.

Let A, B, C, be three persons whose ages are 45, 56, and 68; interest at 4 per cent. here $l = 3$, $m = 32$, $n = 20$, $v = \frac{l+m+n}{3} = 32$ nearly; whence $p = .289$, $1 - p = .711$; then

$$\frac{1-p}{r} = \frac{.711}{.04} = \quad + \quad 17.77$$

$$\frac{p}{lmn} \times \frac{v^3}{r} + \frac{3vvR}{rr} \&c. = 89.62$$

$$- \frac{R^3 + 4R^2 + R}{lmnr^4} = - 92.14$$

$$+ 107.39$$

15.25 the

value of the annuity.

Or thus.

$$\text{Here } x = \frac{43 + 32 + 20}{4} = \frac{95}{4} = 24 \text{ nearly.}$$

And by Tab. II. the present worth of an annuity certain, to continue 24 years (at 4 per cent.) is 15.24 for the value sought.

Or thus.

By Tab. III. the value of the lives of 45 and 56, are 12.68 and 10.47; and (by Cor. 2. Prop. 15.) the value of the joint lives is 7.44; whence (Prop. XIX.) the value of the longest life of AB = $23.15 - 7.44 = 15.71 = Z$; also the life C (68)

(68) = 7.33; and the joint lives of $ZC = 6.24$; and then $Z + C - j. \overline{ZC} = 16.80 = l. ZC = l. ABC$, that is, longest life of $ABC = 16.80$, the value of the annuity.

PROP. XXI. PROB.

To find the value of an annuity of 1l. for the longest of any number of lives, whose ages are given.

This performed by finding successively the value for three lives (as above), then for 4 lives, then for five, and so on, by Prop. VIII. Thus, let A, B, C, D, E, F, &c. be several persons, then,

1. Having found the value for the longest of three lives, as above, put $Z =$ value for the three lives A, B, C; which reckon as a single life. Then find the value of the longest life of ZD , by Prop. XIX. and that will be the value of the longest of the four lives ABCD.

2. Again, put $Z =$ a single life equal to the longest of the four ABCD, and find the value of the longest life of ZE as before, which will be the value of the longest of the five lives ABCDE. It is evident this process may be carried on as far as you will with little trouble.

EXAMPLE.

Let the ages be 45, 56, 68, and 31, interest at 4 per cent. By the last Prop. the value of the longest of the three first lives is $15.25 = Z$; and by Tab. III. the value of a life of 31, is $14.81 = E$. Then $j. ZE = 10.74$, and $E + Z = 30.06$, and $30.06 - 10.74 = 19.32$ the value of the longest of the four lives.

PROP.

PROP. XXII. PROB.

To find the value of an annuity of 1l. to continue as long as any two out of three persons shall be alive.

1 Way.

Let l, m, n , be the complements of life, and x the years certain for the continuance of the annuity. Then the probability of some two being alive and the third dead, (or perhaps all being alive) at the end of x years, will be

$$\frac{l-x \cdot m-x \cdot n-x}{lmn} + \frac{-x \cdot m-x \cdot x}{lmn} + \frac{l-x \cdot n-x \cdot x}{lmn} + \frac{m-x \cdot n-x \cdot x}{lmn}$$

$= \frac{1}{2}$, by the nature of the question, being an equal chance. Therefore

$$lmn - \overbrace{lm+ln+mn}^{x} + \overbrace{l+m+n}^{xx} - x^3 \Bigg\} \\ + \overbrace{lm+ln+mn}^{x} - 2l-2m-2n \cdot xx + 3x^3 \Bigg\} \\ = \frac{1}{2} lmn; \text{ and reduced } 2x^3 - \overbrace{l+m+n}^{x^2} + \frac{1}{2} lmn \\ mn = 0, \text{ and } x^3 - \frac{l+m+n}{2} xx + \frac{1}{4} lmn = 0.$$

And the root extracted will give x , a near approximation.

2. Or thus.

Let A, B, C, be three lives; the three joint lives takes away one life out of ABC. Another life is taken away by finding the reversion for each joint life by Cor. 1. Prop. XI. thus,

$$\begin{aligned} &+ j. ABC = 1 \text{ life extinct.} \\ &+ j. AB - j. ABC = \text{rev. } j. AB \text{ after the 1st life.} \\ &+ j. AC - j. ABC = \text{rev. } j. AC \text{ after the 1st life.} \\ &+ j. BC - j. ABC = \text{rev. } j. BC \text{ after the 1st life.} \\ &\text{Sum } j. AB + j. AC + j. BC - 2j. ABC = \text{total} \\ &\text{value of the annuity for two lives out of three.} \end{aligned}$$

E X A M P.

E X A M P.

Suppose three persons A, B, C, whose ages are 45, 56, and 68; here $l = 43$, $m = 32$, $n = 20$, interest at 4 per cent. Then $x^3 - \frac{92}{2}xx + 6880 = 0$, and the root will be found $= 14.42$; and the present worth of an annuity to continue 14.42 years (by Tab. II.) is 10.79 the value of the two lives.

Or thus.

By Prop. XV. and XVI. we shall find,

$$\begin{array}{rcl} j. AB & = & 8.27 \\ j. AC & = & 6.27 \\ j. BC & = & 5.91 \end{array} \quad . j. ABC = 5.16,$$

$$\begin{array}{r} 20.45 \\ - 2j. ABC = 10.32 \\ \hline \end{array}$$

10.13 the value of the two lives.

P R O P. XXIII. PROB.

To find the value of the reversion of an annuity of l . for one life after another, their ages being given.

1 Way.

Let the two persons be A, F; then by Cor. 1. Prop. X. the value of l . AF — life of A = reversion of F after A. That is, from the present value of the longest liver, subtract the life of the possessor; and the remainder is the value of the reversion due to F.

2. Or thus.

From the present value of the life of the expectant, subtract the present value of their joint lives; and

and the remainder is the value of the expectant's life after the possessor's.

For (by Prop. VII.) $l. AF + j. AF = A + F$,
and $l. AF - A = F - j. AF = \text{reversion}$.

Cor. Let $A = \text{value of the possessor's life}$,
 $F = \text{value of the expectant's life}$,
 $r = \text{interest of } 1l$; then

$$\frac{1 - rA}{1 + 1 - rA} \times F = \text{value of the expectation near}$$

but too great.

For (Cor. 2. Prop. XV.) $F - j. AF = F -$

$$\frac{AF}{A + F - rAF} = \frac{FF - rAFF}{A + F - rAF} = \frac{1 - rA}{A + F - rAF}$$

$$rF = \frac{1 - rA}{\frac{A}{F} + 1 - rA} \times F.$$

EXAMPLE.

Let the possessor's age be 56, and the expectant's 45; interest 4 per cent. then $A = 10.47$, $F = 12.68$;

and $\frac{A}{F} = .83$, and $1 - rA = .58$ and $\frac{1 - rA}{\frac{A}{F} + 1 - rA}$

$\times F = 5.23$, the reversion, but is too much; and will be more exactly done by finding the value of the joint lives by Prop. XV. first way, which will be 8.27; then

$$\begin{array}{rcl} F & = & 12.68 \\ j. AF & = & 8.27 \\ \hline \text{val. annuity} & = & 4.41 \end{array}$$

SCHOL. I.

If it was required to find the value of the annuity for the second person and his heirs. From the value of the perpetuity subtract the value of the possessor's life.

H

SCHOL.

SCHOL. II.

If it was required to find the value of the reversion of an annuity, for the life of any person F, after t years.

From the value of F's life, (Tab. III.) subtract the value of the same life for t years certain (found by Cor. 1. Prop. XIII. annuities); the remainder is the reversion.

PROP. XXIV. PROB.

To find the value of the reversion of an annuity of 1l. for one life after the longest of two, their ages being given.

1 Way.

Let the persons be A, B, F; F being the expectant. Then (by Cor. 3. Prop. X.) value of $l. ABF = l. AB =$ reversion of F after both the lives of AB. That is, if from the value of the longest life of all three, the value of the longest life of the two possessors be taken, the remainder is the value of the reversion of one life after two.

2. Or thus.

The life of F $= j. AF = j. BF + j. ABF =$ value. That is, to the life of the expectant, add the three joint lives; from which subtract the sum of the joint lives of the expectant and each of the possessors; the remainder is the value of the reversion.

For let $Z = l. AB$, then $l. ABF = l. ZF = Z + F - j. ZF = Z + F - j. AF - j. BF + j. ABF$ (by Cor. 4. Prop. IX); and therefore $l. ABF = l. AB = F - j. AF - j. BF + j. ABF$.

EXAMPLE

EXAMPLE.

Let the ages of the possessors be 45 and 68, and the expectant 56; interest 4 per cent. Then by Prop. XIX. and XX.

$$\begin{array}{rcl}
 l. ABF & = & 15.24 \\
 - l. AB & = & 13.74 \\
 \hline
 \text{value — reversion} & & 1.50
 \end{array}$$

Or thus.

$$\begin{array}{rcl}
 F & = & 10.47 \\
 j. ABF & = & 5.16 \\
 \hline
 & & 15.61 \\
 j. AF & = & 8.27 \\
 j. BF & = & 5.90 \\
 \hline
 & & 14.17 \\
 \hline
 \text{reversion} & & 1.44
 \end{array}$$

S C H O L.

If it was required to find the present value of an annuity for a life after two joint lives. By Cor. 3, Prop. XI. we shall have this rule, $F - j. ABF = \text{revers. } F \text{ after } j. AB$. That is, from the value of the expectant's life, subtract the value of the three joint lives; the remainder is the reversion.

PROP. XXV. PROB.

To find the value of the reversion of an annuity of $l.$ for the longest liver of two persons, after the life of one person; all their ages being given.

Let the possessor be A, and expectants F, G. Then (by Cor. 2, Prop. X.) value of $l. AFG -$

H 2

life

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life of A = reversion of l . FG after the life of A. That is, from the value of the longest of the three lives, subtract the possessor's life; the remainder is the value of the reversion.

EXAMPLE.

Let the possessor's age be 56, and the ages of the expectants 45 and 68. Interest 4 per cent. Then (by Prop. XX.) l . ABF = 15.24, and (by Tab. III.) life of A = 10.47,

$$\begin{array}{rcl} l. ABF & = & 15.24 \\ \text{life of A} & = & 10.47 \\ \hline \text{reversion} & & 4.77 \end{array}$$

SCHOOL.

If it was required to find the present value of an annuity, for two joint lives after the life of one person. Then (by Cor. 1. Prop. XI.) j . FG - j . AFG = value of the reversion of j . FG after A. That is, from the value of the joint lives of the two expectants, subtract the value of the three joint lives, the remainder will be the reversion.

PROP. XXVI. PROB.

To find how much the value of an annuity ought to be increased; when the payments are half-yearly or quarterly.

1 Way.

Suppose the year divided into m parts, and d = interest of 1 l . for the m^{th} part of the year, and r = interest of 1 l . for a year, t = number of years the annuity continues. Then,

$$1 + r$$

$$\frac{1 + \frac{r}{100} - 1}{r} = \text{value, for the yearly rents.}$$

$$\text{And } \frac{1 + \frac{r}{100} \frac{m}{d} - 1}{\frac{r}{md}} = \text{value, for the } \frac{1}{m^{\text{th}}} \text{ yearly rents.}$$

$$\text{Cor. } \frac{1 + \frac{r}{2} - 1}{r} = \text{value for half yearly payments.}$$

$$\text{And } \frac{1 + \frac{r}{4} - 1}{r} = \text{value for quarterly payments.}$$

All this follows from Prob. 33, Algebra; augmenting the time and diminishing the interest and the rents, in the supposed ratio,

2. Otherwise.

Let V = value of the yearly payments, then

$$\frac{1 + \frac{r}{m} - 1}{r} V = \text{the value of the half yearly, or}$$

quarterly payments, according as $\frac{1}{m}$ denotes $\frac{1}{2}$ year or $\frac{1}{4}$ the year.

For r is the simple interest for a year, and $1 + \frac{r}{m} - 1$ is the compound interest for all the m^{th} parts of it, through the year,

P R O P. XXVII. PROB.

If A enjoys an annuity of $1l.$ for his life, and at his death has the nomination of a successor B, who is to enjoy the annuity for his life, both their ages being given; to find the value of the successive lives.

1 Way.

Let M be the value of A's life, and N the value of B's, both taken out of Tab. III. let V be the present

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present value of N , for A 's life, then the present value of the successive lives is $= M + V$. That is, find M in Table II, against which you have the year; find this year in Table I, against which is a number (a), by which multiply N ; then $M + aN =$ value of the successive lives.

2. Or thus.

Let M , N , be the values of the two lives, as found in Table III. Also suppose the same values M , N , to be the values of an annuity certain for some numbers of years as m , n , (as in Table II.)

Then (Algebra, Prob. 34, B. II.) $\frac{1 - \frac{1}{R^m}}{r} = M$, and

$1 - \frac{1}{R^n} = rM$, and $\frac{1}{R^n} = 1 - rM$; by the same

reasoning $\frac{1}{R^n} = 1 - rM$. Whence $\frac{1}{R^{m+n}} = 1$

$- rM - rN + rrMN$. Now let $S =$ value of an annuity certain to continue $m + n$ years; then

$\frac{1}{R^{m+n}} = 1 - rS = 1 - rM - rN + rrMN$; and

reduced $S = M + N - rMN$, which is the thing required.

E X A M.

Let the ages be 40 and 60, then $M = 12.09$, $N = 8.83$, at 5 per cent. Then 12.09 found in Table II, gives 19 for the years; and 19 found in Table I, gives .3957, and $.3957 \times 8.83 = 3.49$; then $12.09 + 3.49 = 15.58$, the value required.

Or thus.

Here $MN = 106.7$, and $rMN = 5.34$. And $M + N = 20.92$, and $20.92 - 5.34 = 15.58$, as required.

P R O P.

PROP. XXVIII. PROB.

If A enjoys an annuity for his life, and at his death has the nomination of a successor B; who also, at his death, has the nomination of a successor C, who is to enjoy the annuity for his life. To find the value of the three successive lives.

1 Way.

Let M, N, Q, be the values of the three lives by Table III. Find N in Table II, against which you have the year. Find the year in Table I, against which is a number a , by which multiply Q; then $N + aQ$ is the value of these two, at the time of A's death.

In like manner find M in Table II, against which is the year; find the year in Table I, against which is a number b ; by this multiply $N + aQ$; then $M + bN + abQ =$ value of the three successive lives.

2. Or thus.

Suppose M, N, Q, to be the values of an annuity certain, each for some number of years, as m, n, q . And S the value of an annuity certain for $m+n+q$ years; then S will be the value required. Proceeding as in the last we shall have

$$\frac{1}{R^m} = 1 - rM, \quad \frac{1}{R^n} = 1 - rN, \quad \text{and} \quad \frac{1}{R^q} = 1 - rQ.$$

Whence $\frac{1}{R^{m+n+q}}$ or $1 - rS = \overline{1 - rM} \times \overline{1 - rN} \times \overline{1 - rQ} = 1 - rM - rN - rQ + rrMN + rrMQ + rrNQ - r^3MNQ$. Which being reduced we have $S = M + N + Q - r \times \overline{MN + MQ + NQ + rrMNQ}$.

E x A M.

Let the ages be 30, 40, and 60, at 5 per cent, then $M = 13.19$, $N = 12.09$, $Q = 8.83$. In Tab. II, 12.09 gives 19 for the year, and in Tab. I, 19 gives $.3957 = a$; and $aQ = 3.49$, and $N + aQ = 15.58$.

Again, 13.19 in Table II, gives 22, and 22 in Table I, gives $.3418 = b$, then $M + b \times N + aQ = 18.51$, the value.

Or thus.

$$\begin{array}{rcl}
 M + N + Q & = & 34.11 \\
 + rr MNQ & = & 3.52 \\
 \hline
 & & 37.63 \\
 - r \times MN + MQ + NQ & = & 19.13 \\
 \hline
 \text{The value required} & & 18.50
 \end{array}$$

Here follow the tables of the present value of 1 pound, and of 1 pound annuity certain, for any number of years under 90; and a Table of 1 pound annuity for any age. This last Table is constructed by Prop. XIV, or more easily by Cor. 2, by help of the first and second Tables, which Tables may be proved true, or corrected where wrong, after this manner.

Let p = present worth of 1*l.* for any time t .

P = present worth of an annuity of 1*l.*

a = amount of 1*l.*

A = amount of 1*l.* annuity, all for t years.

r = rate of interest of 1*l.* for a year. All at compound interest.

$R = 1 + r$, the amount of 1*l.* for a year.
Then,

$$ap = 1.$$

$$rA = a - 1.$$

$$\frac{1}{r} = \text{perpetuity.}$$

$A + a =$ amount of 1*l.* annuity for $t + 1$ years.

$$P = \frac{A}{a} = pA = \frac{1-p}{r} = \frac{1}{r} - \frac{1}{rR^t}.$$

Sum of n terms in Table I, = the n^{th} term in Table II, by the nature of annuities.

The sum of $n - 1$ terms in Table II = $n \times$ num. in Table III, whose age is $88 - n$. See Prop. XIV.

These equations shew the relation of the numbers in the two first Tables, and how one may be constructed from the other.

It appears by the Table Prop. I, that the numbers are irregular as far as 8 years of age. And therefore the numbers in Table III, under 8 are supplied by another rule, which is this. In the Table Prop. I, let $a, b, c, \&c.$ to f , be the numbers (or persons living) against the ages $A, A + 1, A + 2, \&c.$ to $A + y$. And let $V =$ present worth of an annuity for the age $A + y$. Then the present

worth of an annuity for the age A is = $\frac{1}{a} \times$ into $\frac{b}{R}$

+ $\frac{c}{R^2} + \frac{d}{R^3} \&c.$ to $\frac{f}{R^y} \times$ into $1 + V$. Or to find

them number by number, take only one term, then

$\frac{b}{aR} \times 1 + V =$ present worth.

TABLE

ANNUITIES.

TABLE I. *The present value of 1 pound, due any number of years hence. At 3 per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	.97087	31	.39999	61	.16479
2	.94259	32	.38834	62	.15999
3	.91514	33	.37702	63	.15533
4	.88848	34	.36604	64	.15080
5	.86261	35	.35538	65	.14641
6	.83748	36	.34503	66	.14215
7	.81309	37	.33498	67	.13801
8	.78941	38	.32522	68	.13399
9	.76642	39	.31575	69	.13008
10	.74409	40	.30656	70	.12630
11	.72242	41	.29763	71	.12262
12	.70138	42	.28896	72	.11905
13	.68095	43	.28054	73	.11558
14	.66112	44	.27237	74	.11221
15	.64186	45	.26444	75	.10894
16	.62317	46	.25674	76	.10577
17	.60502	47	.24926	77	.10269
18	.58739	48	.24200	78	.09970
19	.57028	49	.23495	79	.09679
20	.55367	50	.22811	80	.09398
21	.53755	51	.22146	81	.09124
22	.52189	52	.21501	82	.08858
23	.50669	53	.20875	83	.08600
24	.49193	54	.20267	84	.08350
25	.47760	55	.19677	85	.08106
26	.46369	56	.19103	86	.07870
27	.45019	57	.18547	87	.07641
28	.43708	58	.18007	88	.07418
29	.42434	59	.17482	89	.07203
30	.41199	60	.16973	90	.06993

TABLE I. The present value of 1 pound, at $3\frac{1}{2}$ per cent. compound interest.

age.	value.	age.	value.	age.	value.
1	.96618	31	.34423	61	.12264
2	.93351	32	.33259	62	.11849
3	.90194	33	.32134	63	.11449
4	.87144	34	.31047	64	.11061
5	.84197	35	.29998	65	.10587
6	.81350	36	.28983	66	.10326
7	.78599	37	.28003	67	.09977
8	.75941	38	.27056	68	.09639
9	.73373	39	.26141	69	.09313
10	.70892	40	.25257	70	.08998
11	.68494	41	.24403	71	.08694
12	.66178	42	.23578	72	.08400
13	.63940	43	.22780	73	.08116
14	.61778	44	.22010	74	.07842
15	.59689	45	.21265	75	.07576
16	.57670	46	.20547	76	.07320
17	.55720	47	.19852	77	.07073
18	.53836	48	.19180	78	.06833
19	.52015	49	.18532	79	.06602
20	.50256	50	.17905	80	.06379
21	.48557	51	.17300	81	.06163
22	.46915	52	.16715	82	.05955
23	.45328	53	.16149	83	.05754
24	.43796	54	.15603	84	.05559
25	.42315	55	.15076	85	.05371
26	.40884	56	.14566	86	.05189
27	.39501	57	.14073	87	.05014
28	.38165	58	.13597	88	.04844
29	.36875	59	.13138	89	.04681
30	.35628	60	.12693	90	.04522

TABLE I. *The present value of 1 pound, at 4 per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	.96154	31	.29646	61	.09140
2	.92455	32	.28506	62	.08789
3	.88899	33	.27409	63	.08451
4	.85480	34	.26355	64	.08126
5	.82193	35	.25341	65	.07813
6	.79031	36	.24367	66	.07513
7	.75992	37	.23430	67	.07224
8	.73069	38	.22528	68	.06946
9	.70259	39	.21662	69	.06679
10	.67556	40	.20829	70	.06422
11	.64958	41	.20028	71	.06175
12	.62460	42	.19257	72	.05937
13	.60057	43	.18517	73	.05709
14	.57747	44	.17804	74	.05489
15	.55526	45	.17120	75	.05278
16	.53391	46	.16461	76	.05075
17	.51337	47	.15828	77	.04880
18	.49363	48	.15219	78	.04692
19	.47464	49	.14634	79	.04512
20	.45638	50	.14071	80	.04338
21	.43883	51	.13530	81	.04171
22	.42195	52	.13010	82	.04011
23	.40572	53	.12509	83	.03857
24	.39012	54	.12028	84	.03708
25	.37512	55	.11565	85	.03566
26	.36069	56	.11121	86	.03429
27	.34682	57	.10693	87	.03297
28	.33348	58	.10282	88	.03179
29	.32065	59	.09886	89	.03048
30	.30832	60	.09506	90	.02931

TABLE I. *The present value of 1 pound, at $4\frac{1}{2}$ per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	.95694	31	.25550	61	.06822
2	.91573	32	.24450	62	.06528
3	.87630	33	.23397	63	.06247
4	.83856	34	.22389	64	.05978
5	.80245	35	.21425	65	.05720
6	.76789	36	.20503	66	.05474
7	.73483	37	.19620	67	.05238
8	.70318	38	.18775	68	.05013
9	.67290	39	.17966	69	.04797
10	.64393	40	.17193	70	.04590
11	.61620	41	.16452	71	.04393
12	.58966	42	.15744	72	.04204
13	.56427	43	.15066	73	.04022
14	.53997	44	.14417	74	.03849
15	.51672	45	.13796	75	.03683
16	.49447	46	.13202	76	.03525
17	.47317	47	.12634	77	.03373
18	.45280	48	.12090	78	.03228
19	.43330	49	.11569	79	.03089
20	.41464	50	.11071	80	.02956
21	.39679	51	.10594	81	.02829
22	.37970	52	.10138	82	.02707
23	.36335	53	.09701	83	.02590
24	.34770	54	.09284	84	.02479
25	.33273	55	.08884	85	.02372
26	.31840	56	.08501	86	.02270
27	.30469	57	.08135	87	.02172
28	.29157	58	.07785	88	.02078
29	.27901	59	.07450	89	.01989
30	.26700	60	.07129	90	.01903

TABLE I. *The present worth of 1 pound, at per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	.95238	31	.22036	61	.05098
2	.90703	32	.20986	62	.04856
3	.86384	33	.19987	63	.04624
4	.82270	34	.19035	64	.04404
5	.78352	35	.18129	65	.04194
6	.74621	36	.17266	66	.03995
7	.71068	37	.16443	67	.03805
8	.67684	38	.15660	68	.03623
9	.64461	39	.14915	69	.03451
10	.61391	40	.14204	70	.03286
11	.58468	41	.13528	71	.03130
12	.55684	42	.12884	72	.02981
13	.53032	43	.12270	73	.02839
14	.50507	44	.11686	74	.02704
15	.48102	45	.11130	75	.02575
16	.45811	46	.10600	76	.02452
17	.43630	47	.10095	77	.02336
18	.41552	48	.09614	78	.02224
19	.39573	49	.09156	79	.02118
20	.37689	50	.08720	80	.02018
21	.35894	51	.08305	81	.01921
22	.34185	52	.07909	82	.01830
23	.32557	53	.07533	83	.01743
24	.31007	54	.07174	84	.01660
25	.29530	55	.06832	85	.01581
26	.28124	56	.06507	86	.01505
27	.26785	57	.06197	87	.01434
28	.25509	58	.05902	88	.01366
29	.24294	59	.05621	89	.01300
30	.23138	60	.05353	90	.01239

TABLE II. *The present value of an annuity of 1 pound per annum, at 3 per cent, compound interest.*

years.	value.	years.	value.	years.	value.
1	0.97087	31	20.00043	61	27.84035
2	1.91347	32	20.38876	62	28.00034
3	2.82861	33	20.76579	63	28.15567
4	3.71710	34	21.13184	64	28.30648
5	4.57971	35	21.48722	65	28.45289
6	5.41719	36	21.83225	66	28.59504
7	6.23028	37	22.16723	67	28.73305
8	7.01969	38	22.49246	68	28.86704
9	7.78611	39	22.80821	69	28.99713
10	8.53020	40	23.11477	70	29.12342
11	9.25262	41	23.41240	71	29.24604
12	9.95400	42	23.70136	72	29.36509
13	10.63495	43	23.98190	73	29.48067
14	11.29607	44	24.25427	74	29.59288
15	11.93793	45	24.51871	75	29.70183
16	12.56110	46	24.77545	76	29.80760
17	13.16612	47	25.02471	77	29.91029
18	13.57351	48	25.26671	78	30.00999
19	14.32380	49	25.50166	79	30.10679
20	14.87747	50	25.72976	80	30.20076
21	15.41502	51	25.95123	81	30.29200
22	15.93692	52	26.16624	82	30.38059
23	16.44361	53	26.37499	83	30.46659
24	16.93554	54	26.57766	84	30.55009
25	17.41315	55	26.77443	85	30.63115
26	17.87684	56	26.96546	86	30.70986
27	18.32703	57	27.15093	87	30.78627
28	18.76411	58	27.33100	88	30.86045
29	19.18845	59	27.50583	89	30.93248
30	19.60044	60	27.67556	90	30.00241

TABLE

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TABLE II. *The present value of an annuity of pound per annum, at $3\frac{1}{2}$ per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.96618	31	18.73627	61	25.06738
2	1.89969	32	19.06886	62	25.18587
3	2.80164	33	19.39021	63	25.30036
4	3.67308	34	19.70068	64	25.41098
5	4.51505	35	20.00066	65	25.51785
6	5.32855	36	20.29049	66	25.62112
7	6.11454	37	20.57052	67	25.72089
8	6.87395	38	20.84109	68	25.81727
9	7.60769	39	21.10250	69	25.91041
10	8.31660	40	21.35507	70	26.00040
11	9.00155	41	21.59910	71	26.08734
12	9.66333	42	21.83488	72	26.17134
13	10.30274	43	22.06269	73	26.25251
14	10.92052	44	22.28279	74	26.33092
15	11.51741	45	22.49545	75	26.40669
16	12.09412	46	22.70092	76	26.47989
17	12.65132	47	22.89944	77	26.55062
18	13.18968	48	23.09124	78	26.61896
19	13.70984	49	23.27656	79	26.68498
20	14.21240	50	23.45562	80	26.74878
21	14.69797	51	23.62861	81	26.81041
22	15.16712	52	23.79576	82	26.86996
23	15.62041	53	23.95726	83	26.92770
24	16.05837	54	24.11329	84	26.98309
25	16.48151	55	24.26405	85	27.03681
26	16.89035	56	24.40971	86	27.08870
27	17.28536	57	24.55045	87	27.13884
28	17.66702	58	24.68642	88	27.18729
29	18.03577	59	24.81780	89	27.23399
30	18.39204	60	24.94473	90	27.27032

TABLE II. *The present value of an annuity of 1 pound per annum, at 4 per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.96154	31	17.58849	61	22.71489
2	1.88609	32	17.87355	62	22.80278
3	2.77509	33	18.14764	63	22.88729
4	3.62989	34	18.41120	64	22.96855
5	4.45182	35	18.66461	65	23.04668
6	5.24214	36	18.90828	66	23.12181
7	6.00205	37	19.14258	67	23.19405
8	6.73274	38	19.36786	68	23.26351
9	7.43533	39	19.58448	69	23.33030
10	8.11089	40	19.79277	70	23.39451
11	8.76048	41	19.99305	71	23.45626
12	9.38507	42	20.18563	72	23.51564
13	9.98565	43	20.37079	73	23.57273
14	10.56312	44	20.54884	74	23.62762
15	11.11839	45	20.72004	75	23.68041
16	11.65229	46	20.88465	76	23.73116
17	12.16567	47	21.04293	77	23.77996
18	12.65930	48	21.19513	78	23.82689
19	13.13394	49	21.34147	79	23.87201
20	13.59032	50	21.48218	80	23.91539
21	14.02916	51	21.61748	81	23.95711
22	14.45111	52	21.74758	82	23.99722
23	14.85684	53	21.87267	83	24.03579
24	15.24696	54	21.99296	84	24.07287
25	15.62208	55	22.10861	85	24.10853
26	15.98277	56	22.21982	86	24.14282
27	16.32958	57	22.32675	87	24.17579
28	16.66306	58	22.42957	88	24.20749
29	16.98371	59	22.52843	89	24.23797
30	17.29203	60	22.62349	90	24.26728

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TABLE II. *The present value of an annuity of pound per annum, at $4\frac{1}{2}$ per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.95694	31	16.54439	61	20.70625
2	1.87267	32	16.78889	62	20.77152
3	2.74896	33	17.02286	63	20.83399
4	3.58752	34	17.24676	64	20.89377
5	4.38998	35	17.46101	65	20.95098
6	5.15787	36	17.66604	66	21.00571
7	5.89270	37	17.86224	67	21.05811
8	6.59588	38	18.04999	68	21.10824
9	7.26879	39	18.22965	69	21.15621
10	7.91272	40	18.40158	70	21.20211
11	8.52892	41	18.56611	71	21.24604
12	9.11858	42	18.72355	72	21.28788
13	9.68285	43	18.87421	73	21.32783
14	10.22282	44	19.01838	74	21.36679
15	10.73954	45	19.15635	75	21.40393
16	11.23401	46	19.28837	76	21.43888
17	11.70719	47	19.41471	77	21.47261
18	12.15999	48	19.53561	78	21.50490
19	12.59329	49	19.65130	79	21.53578
20	13.00793	50	19.76201	80	21.56534
21	13.40472	51	19.86795	81	21.59363
22	13.78442	52	19.96933	82	21.62070
23	14.14777	53	20.06634	83	21.64660
24	14.49548	54	20.15918	84	21.67139
25	14.82821	55	20.24802	85	21.69511
26	15.14661	56	20.33303	86	21.71781
27	15.45130	57	20.41439	87	21.73953
28	15.74287	58	20.49223	88	21.76031
29	16.02189	59	20.56673	89	21.78020
30	16.28889	60	20.63802	90	21.79924

TABLE II. *The present worth of an annuity of 1 pound per annum, at 5 per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.95238	31	15.59281	61	18.98027
2	1.85941	32	15.80268	62	19.02883
3	2.72325	33	16.00255	63	19.07508
4	3.54595	34	16.19290	64	19.11912
5	4.32948	35	16.37419	65	19.16107
6	5.07569	36	16.54685	66	19.20102
7	5.78637	37	16.71129	67	19.23906
8	6.46321	38	16.86789	68	19.27530
9	7.10782	39	17.01704	69	19.30981
10	7.72173	40	17.15908	70	19.34268
11	8.30641	41	17.29437	71	19.37398
12	8.86325	42	17.42321	72	19.40379
13	9.39357	43	17.54591	73	19.43218
14	9.89864	44	17.66277	74	19.45922
15	10.37966	45	17.77407	75	19.48497
16	10.83777	46	17.88007	76	19.50949
17	11.27406	47	17.98101	77	19.53285
18	11.68959	48	18.07716	78	19.55510
19	12.08532	49	18.16872	79	19.57628
20	12.46221	50	18.25592	80	19.59646
21	12.82115	51	18.33898	81	19.61568
22	13.16300	52	18.41807	82	19.63398
23	13.48857	53	18.49340	83	19.65141
24	13.79864	54	18.56514	84	19.66801
25	14.09394	55	18.63347	85	19.68381
26	14.37518	56	18.69854	86	19.69887
27	14.64303	57	18.76052	87	19.71321
28	14.89813	58	18.81954	88	19.72687
29	15.14107	59	18.87575	89	19.73987
30	15.37245	60	18.92929	90	19.75226

TABLE III. *The present value of annuities upon life of any age; at 3 per cent.*

age.	value.	age.	value.	age.	value.
0	14.53				
1	16.63	31	16.97	61	10.02
2	18.63	32	16.79	62	9.72
3	19.64	33	16.61	63	9.42
4	20.07	34	16.43	64	9.11
5	20.29	35	16.25	65	8.80
6	20.40	36	16.06	66	8.47
7	20.42	37	15.86	67	8.14
8	20.38	38	15.66	68	7.80
9	20.26	39	15.46	69	7.45
10	20.12	40	15.26	70	7.10
11	20.00	41	15.05	71	6.75
12	19.87	42	14.84	72	6.39
13	19.74	43	14.62	73	6.02
14	19.60	44	14.40	74	5.64
15	19.45	45	14.18	75	5.25
16	19.31	46	13.95	76	4.85
17	19.17	47	13.72	77	4.44
18	19.03	48	13.49	78	4.03
19	18.89	49	13.25	79	3.62
20	18.76	50	13.01	80	3.21
21	18.61	51	12.76	81	2.79
22	18.46	52	12.50	82	2.36
23	18.31	53	12.24	83	1.91
24	18.15	54	11.98	84	1.46
25	18.00	55	11.72	85	1.00
26	17.84	56	11.44	86	0.50
27	17.67	57	11.16	87	0.00
28	17.50	58	10.88		
29	17.32	59	10.60		
30	17.15	60	10.32		

TABLE III. *The present value of annuities upon life of any age; at $3\frac{1}{2}$ per cent.*

age.	value.	age.	value.	age.	value.
0	13.28				
1	15.21	31	15.84	61	9.64
2	17.02	32	15.69	62	9.36
3	17.95	33	15.53	63	9.07
4	18.35	34	15.37	64	8.78
5	18.56	35	15.20	65	8.49
6	18.67	36	15.03	66	8.18
7	18.70	37	14.86	67	7.86
8	18.68	38	14.69	68	7.54
9	18.58	39	14.52	69	7.22
10	18.48	40	14.34	70	6.90
11	18.37	41	14.16	71	6.57
12	18.26	42	13.97	72	6.23
13	18.15	43	13.78	73	5.87
14	18.04	44	13.59	74	5.51
15	17.93	45	13.40	75	5.14
16	17.82	46	13.20	76	4.76
17	17.70	47	12.99	77	4.37
18	17.58	48	12.78	78	3.97
19	17.46	49	12.57	79	3.57
20	17.34	50	12.35	80	3.16
21	17.22	51	12.13	81	2.74
22	17.10	52	11.90	82	2.31
23	16.98	53	11.67	83	1.87
24	16.85	54	11.44	84	1.42
25	16.73	55	11.20	85	0.98
26	16.58	56	10.96	86	0.49
27	16.43	57	10.71	87	0.00
28	16.28	58	10.45		
29	16.13	59	10.18		
30	15.98	60	9.91		

ANNUITIES.

TABLE III. *The present value of annuities, upon a life of any age; at 4 per cent.*

age.	value.	age.	value.	age.	value.
0	12.23				
1	13.98	31	14.81	61	9.27
2	15.66	52	14.68	62	9.01
3	16.51	33	14.55	63	8.74
4	16.90	34	14.41	64	8.47
5	17.09	35	14.27	65	8.20
6	17.20	36	14.13	66	7.92
7	17.24	37	13.98	67	7.63
8	17.23	38	13.83	68	7.33
9	17.14	39	13.68	69	7.02
10	17.06	40	13.52	70	6.71
11	16.97	41	13.36	71	6.39
12	16.88	42	13.19	72	6.07
13	16.79	43	13.02	73	5.73
14	16.70	44	12.85	74	5.38
15	16.60	45	12.68	75	5.03
16	16.50	46	12.50	76	4.67
17	16.40	47	12.32	77	4.39
18	16.30	48	12.13	78	3.91
19	16.20	49	11.94	79	3.52
20	16.10	50	11.75	80	3.12
21	15.99	51	11.55	81	2.71
22	15.88	52	11.34	82	2.28
23	15.77	53	11.13	83	1.85
24	15.66	54	10.92	84	1.40
25	15.55	55	10.70	85	0.95
26	15.43	56	10.47	86	0.48
27	15.31	57	10.24	87	0.00
28	15.19	58	10.01		
29	15.07	59	9.77		
30	14.94	60	9.53		

TABLE III. *The present value of annuities, upon life of any age; at $4\frac{1}{2}$ per cent.*

age.	value.	age.	value.	age.	value.
0	11.31				
1	12.92	31	13.91	61	8.93
2	14.47	32	13.79	62	8.69
3	15.26	33	13.67	63	8.45
4	15.63	34	13.55	64	8.20
5	15.82	35	13.43	65	7.94
6	15.92	36	13.31	66	7.67
7	15.97	37	13.18	67	7.40
8	15.96	38	13.05	68	7.12
9	15.89	39	12.91	69	6.83
10	15.82	40	12.77	70	6.53
11	15.75	41	12.63	71	6.23
12	15.68	42	12.49	72	5.92
13	15.60	43	12.34	73	5.60
14	15.52	44	12.19	74	5.27
15	15.44	45	12.03	75	4.93
16	15.36	46	11.87	76	4.58
17	15.28	47	11.71	77	4.22
18	15.19	48	11.54	78	3.85
19	15.10	49	11.37	79	3.47
20	15.01	50	11.19	80	3.07
21	14.92	51	11.01	81	2.67
22	14.83	52	10.82	82	2.26
23	14.74	53	10.64	83	1.83
24	14.64	54	10.44	84	1.39
25	14.54	55	10.24	85	0.94
26	14.44	56	10.04	86	0.48
27	14.34	57	9.83	87	0.00
28	14.24	58	9.61		
29	14.13	59	9.39		
30	14.02	60	9.16		

ANNUITIES.

TABLE III. *The present value of annuities, upon a life of any age; at 5 per cent.*

age	value.	age.	value.	age.	value.
0	10.52				
1	12.01	31	13.09	61	8.62
2	13.44	32	12.99	62	8.40
3	14.18	33	12.89	63	8.17
4	14.52	34	12.78	64	7.93
5	14.70	35	12.67	65	7.68
6	14.81	36	12.56	66	7.43
7	14.85	37	12.45	67	7.17
8	14.85	38	12.33	68	6.91
9	14.80	39	12.21	69	6.64
10	14.73	40	12.09	70	6.36
11	14.67	41	11.96	71	6.07
12	14.61	42	11.83	72	5.77
13	14.55	43	11.70	73	5.46
14	14.48	44	11.57	74	5.15
15	14.41	45	11.43	75	4.83
16	14.34	46	11.29	76	4.49
17	14.27	47	11.14	77	4.14
18	14.20	48	10.99	78	3.78
19	14.13	49	10.84	79	3.41
20	14.05	50	10.68	80	3.03
21	13.97	51	10.52	81	2.64
22	13.89	52	10.35	82	2.24
23	13.81	53	10.18	83	1.82
24	13.73	54	10.00	84	1.36
25	13.64	55	9.82	85	0.94
26	13.55	56	9.63	86	0.47
27	13.46	57	9.44	87	0.00
28	13.37	58	9.24		
29	13.28	59	9.04		
30	13.19	60	8.83		

A R T. III.

of the Institution of Societies ; or the Calculation of Annuities to be paid in any Society, instituted for the Benefit of old People, &c.

THE Calculation of Annuities for persons entering into Societies, depends upon the Laws of Chance, and also upon the Rules of Annuities delivered in the last Article. But some questions of this sort are so intricate, involve so many conditions, and require so many calculations, as to render them extremely difficult and laborious to perform. In such cases we must be content to make use of such approximations as will shorten the work, and come near the truth. Every society ought to be established upon such principles, that it may be able to continue for ever ; and therefore all the conditions must be duly adjusted to this end. And although, by the variableness and chances of things, the fund may sometimes lose ; it will at other times gain as much, and so will never fail. Several Problems of this sort here follow.

A X I O M I.

The present value of what is paid to the fund, must be at least equal to the present value of what is received by the annuitants.

A X I O M II.

It is the same thing whether a number of people all enter at once, or some at one time and some at another.

P R O B.

PROB. I.

A person whose age is a , puts a sum of money s , into the hands of some society, which sum is to lie dead for y years, after which he is to receive an annuity for his life. To find that annuity, reckoning at i per cent.

Let x = annuity, m the complement of a , and n the complement of $a+y$. Then suppose (Ax. 2.) there are m persons, one will die every year; and in y years y of them will die, and receive no annuity. Now the money paid is $=ms$, and let the value of a life whose age is $a+y$ be $=v$, then nv = money received by the remaining n people, this is the sum of all the annuities. Find p the present worth of nv due y years hence; then (Ax. 1.) $px = ms$, and $x = \frac{ms}{p}$, the annuity of each person.

E X A M.

Suppose a life of $42 = a$, and $y = 10$ years, $s = 50$ l. then $m = 46$, $n = 36$; and by Table III. a life of $a+y$ or $52 = 11.34 = v$, and $nv = 36 \times 11.34 = 408.38$; then the value of this 10 years sooner is (by Table I.) $408.38 \times .6755 = 275.8 = p$, and $ms = 46 \times 50 = 2300$, and $x = \frac{2300}{275.8} = 8.33$, the annuity.

Therefore, if by this method the annuity for every age be computed, a table will be had, which will shew the annuity for any age.

Cor. 1. If $y = 0$, and $s = 11.34$; then $x = 1$, for the annuity.

Cor.

Cor. 2. If x be given, and the fine s be required,

$$ben s = \frac{px}{m}.$$

SCHOL.

If no interest be reckoned, then the money paid will be 2300 as before. But the money received will be $1 + 2 + 3 + 4$ &c. to 35 = $\frac{35 \times 36}{2} = 630$;

whence $630x = 2300$, and $x = \frac{2300}{630} = 3.65$.

Therefore the greatest part of the annuity arises from the interest.

PROB. II.

A person whose age is a , pays no fine, but pays b pounds a year to some society for y years. To find what annuity he must have afterwards for his life, at 4. per cent.
(London Union Society.)

Let x = annuity, m = compl. of a , n = comp. of $a + y$. Then suppose m persons to enter, y of them will die in y years.

1. Then the money paid by the dead men y , is to be found by Cor. 2. Prop. XIV. Annuities, putting y for n . Or (which is the same thing,) seek the age $88 - y$ in Table III. against which is v , the value of that life; this multiplied by y , that is vy = money paid by the y dead men, at 1l. annuity. And bvy = whole that is paid by these dead men.

Then to find what is paid by the remaining n persons; here each of these persons pays an annuity certain for y years. In Table II. against y find p the present worth, then bpn = sum paid by the n persons. Whence $bvy + bpn$ = whole sum paid by all.

2. The

2. The money received is the annuity x , for 1, 2, 3, 4, &c. to $n-1$ years $= nq$, q being the present value of a life of the age $a+y$. Then $qnx =$ money received after y years. Find t in Table I. the present worth of 1*l.* due y years hence. Then $qnx t =$ present worth of all the money received; whence (Ax. 1.) $qnx t = bvy + bpn$, and $x = \frac{bvy + bpn}{qnt}$ the annuity sought.

E X A M.

Let the age be $42 = a$, $b = 1$, $y = 10$; then $m = 46$, $n = 36$.

The age $88 - y$, or 78 in Table III. gives $v = 3.91$, then $vy = 39.10$, and $bvy = 39.10$.

Again, Table II. against 10 is $8.11 = p$, and then $bpn = 291.96$ and $bvy + bpn = 331.06$, all the money paid.

Then $a + y = 52$. and against 52 (Table III.) is $11.34 = q$, and $t = .6755$ by Table I. whence $qnt = 275.7$, and $x = \frac{331.06}{275.7} = 1.20$, the annuity received.

S C H O L.

By the help of these two problems, that case may be solved, when both a fine and an annuity is to be paid into the fund; and the annuity required, for the surviving members. As in the last example, let $f =$ fine or præmium that each pays, then $46f$ or mf is to be added to the money paid, and we shall have $x = \frac{mf + bvy + bpn}{qnt}$. And if the an-

nunity received (x) be given, the annuity (b) to be paid, may be found. (York Universal Society.)

P R O B.

PROB. III.

A person of the age a , enters into a society (instituted for young people,) and pays b pounds a year for y years, and then lies dead for z years. To find what annuity, he shall have afterwards for his life, at 4 per cent.

Let x annuity to be received, $m = \text{comp. } a$, $n = \text{comp. } a + y$, $w = \text{comp. } a + y + z$. Then suppose m persons in the beginning; then y of them will die in y years, and $y + z$ in $y + z$ years.

1. The money paid by the dead men is found thus, seek the number $88 - y$ in Table III. against which is v , then $bvy = \text{waste money paid by the } y \text{ dead people.}$

For the other n persons, against y in Table II. find p ; then $bpn = \text{money paid by these } n \text{ persons,}$ and the whole is $= bvy + bpn$.

2. The money received is the annuity x for 1, 2, 3, &c. to $w - 1$ years; and this is $= wq$, where q is the value of a life whose age is $a + y + z$, found in Table III. Therefore $qwx = \text{money received after } y + w \text{ years.}$ In Table I. find t the present worth of 1 $l.$ due $y + z$ years hence: then $qwx t = \text{present worth of all the money received for the annuitants.}$ Whence $qwx t = bvy + bpn$, and $x = \frac{bvy + bpn}{qwt}$, the annuity for each person.

EXAMPLE.

Suppose the age be $40 = a$, $b = 5$, $y = 7$, $z = 6$, and $m = 48$, $n = 41$, $w = 35$, $88 - y = 81$, $a + y + z = 53$, $y + z = 13$.

1. Against 81 (Table III.) is $2.71 = v$, then $bvy = 94.85$ the dead money.

Against

Against 7 (Table II.) find $6.002 = p$, then $bpn = 1230.41$; and the whole sum $bvy + bpn = 1325.26$.

2. Against 53 (Table III.) is $11.13 = q$, and against 13 (Table I.) is $.6005 = t$. Then $qwt = 233.9$; whence $x = \frac{1325.2}{233.9} = 5.67$, the annuity.

PROB IV.

A person whose age is a , enters into society, and pays f for a fine, and the annuity certain of b pounds a year, for y years. And afterwards a person of the age c (nominated at first,) is to receive an annuity for his own life. But if the nominee dies within the term, the payment ceases. To find the annuity, at 4 per cent. (York Union Society.)

1. Let $x =$ annuity, $m =$ comp. a , $n =$ comp. $c + y$. Suppose m contributors at first, then $mf =$ sum of all the fines: and then y nominees will die in y years; and then the y contributors will pay the annuity b for 1, 2, 3 — to $y - 1$ years $= byA$, where $A =$ number in Table III. against the age $88 - y$.

The other n contributors will pay nvb , where v is the number in Table II. against y . Therefore $mf + byA + bnv =$ whole money paid.

2. The n nominees remaining will receive the annuity x for 1, 2, 3 — to $n - 1$ years $= tpx$, where $t =$ number against y in Table I. And p is the number against $c + y$ in Table III. Then $tpnx = mf + byA + bnv$, and $x = \frac{mf + byA + bnv}{ptn}$.

E X A M.

Let $a=40$, $f=10$, $b=5$, $y=17$, $c=30$, $m=48$, $n=41$. Then $mf=480$. Against 71 ($88-y$) in Table III. is $6.39=A$, then $byA=543.15$.

Against 17 (Table II.) is $12.165=v$, then $bmv=2493.82$; the sum is 3516.97 .

Against 47 ($c+y$) in Table III. is $12.32=p$. And against 17 in Table I. is $.5134=t$. Whence

$$ptn=259.33, \text{ and } x = \frac{3516.97}{259.33} = 13.56.$$

P R O B. V.

A man and his wife enter into a society, on these terms; that every man shall pay a præmium or admission fine f , and b pounds a year, till one of them be dead. After the husband's death, the wife (if living) is to receive an annuity for her life. To find this annuity; supposing money at 4 per cent.

(Laudable Society.)

Let a be the man's age, c the woman's, x = annuity.

This Prob. is solved by finding the value of the reversion of an annuity for one life after another, by Prop. XXIII. Annuities. Therefore (by Prop. XV. ib.) find G the joint lives of the two persons. And by Table III. find W the value of the wife's life; then $W-G$ = value of the reversion, at 1l. per annum.

It is evident, that $f+bG$ (the value of the joint lives) is all the money paid by the husband; and the reversion $W-G \times x$ is all the money received by the wife. Therefore $W-G \times x = bG + f$,

$$\text{and } x = \frac{f+bG}{W-G}, \text{ the annuity.}$$

If

If both husband and wife are to have the chance then it is evident, the annuity will be but half so much.

E X A M.

Let $a=40$, $c=30$, $f=10$, $b=5$, then the value of the joint lives is $10.28=G$, and $W=14.94$ then $x = \frac{61.40}{4.66} = 13.18$, the annuity.

If a and $c = 40$, f and $b = 1$. Then $W = 13.52$, $G = 9.63$, (by Cor. 3. Prop. 15. annuities,) whence $x = \frac{10.63}{3.89} = 2.73$: but finding $G = 10.12$ (by Rule 1. Prop. 15.) then x will come out 3.27, more exact.

If women live longer than men, or there be more widows than widowers, as some assert; suppose as 3 to 2; then instead of $W - G$, take

$$\overline{W - G} \times 1 + \frac{3 - 2}{3 + 2} \text{ or } \overline{W - G} \times 1 \frac{1}{5}.$$

S C H O L.

If it be required to find how many contributors and annuitants there are in any year. It may be found thus, when the lives are equal; as suppose they are both 40. Here 48 is the comp. life; therefore suppose 48 persons of either sort, then one will die constantly in a year. Let $n = 48$, and $d =$ men or women dead in d years, and l (or $n - d$) = those living.

Since the living or dead belong promiscuously to the whole number, therefore find by proportion (by Prop. VII. *Laws of Chance*) how many living men belong to the living women. That part will be $\frac{l}{n}$; that is, out of 48, $\frac{l}{n}$ men have wives, and therefore that is the number of contributors at d years.

Again,

Again, find how many living women there are belonging to the dead men; and $\frac{d}{n} \times l$ is the number, and this is the number of widows. Whence after d years, there are $\frac{l}{n}$ contributors, and $\frac{dl}{n}$ annuitants, let d be what it will: and consequently the number of contributors will then be to the number of widows, as l to d .

And from hence, by help of Table I. one may compute what the widow's annuity will be, if any she will be at the pains to do it, after the following manner. Let the annuity paid by each contributor be $1l$. Then make a column of all years to 48, against which put down for each year $\frac{l}{n} \times$ by the correspondent number of Table I. The sum of all these 48 numbers added to the præmium, will be the present worth of all the money paid, during the whole term.

Again, put down against these, in another column, for each year $\frac{ld}{n} \times$ by the correspondent number of Table I. (as in the former column,) and the sum of all these numbers $\times d$ by x , will be the present worth of all the money received by the widows. And therefore the common annuity x will be known by division, and this is at $1l$. annual contribution.

Pursuing this method for the age of 40 for man and wife, and $1l$. fine and annuity; I found the widow's annuity $3l.27$; agreeable to what was shewn before.

And hence it follows, that in this scheme, the number of contributors ($\frac{l}{n}$) is greatest at first, and continually decreases to the last.

K

And

And the number of widows $\left(\frac{dl}{n}\right)$, is greater when $d=l$, that is, when d or $l = \frac{1}{2}n$, and then the widows and contributors are equal; and each number is $= \frac{1}{4}n$. Afterwards they continually decrease to 0.

PROB. VI.

A man and his wife of the same age (between 20 and 45) enter into society; and pay 2l. admission fine and 1l a year for his life. And afterwards the widow is to enjoy an annuity for her life; which will be more or less, according as her husband has contributed a longer or a shorter time; that is, for 1, 7, or 13 years. To find the annuity to be paid in each of these intervals, at 4 per cent. (London Annuity Society.)

The age of the persons must be given, or else the Prob. is unlimited. Suppose the age 40, the complement 48.

Let a = age, y = time of continuance, m = comp. a , n = comp. $a+y$, f = fine, x = annuity, v = value of an annuity certain for y years, Table II. p = value of the life $a+y$, Table III. t = number against y , Table I.

Suppose there be m contributors at first; at the end of y years, there will only be n contributors; therefore at a mean, through the time y , there will be $\frac{m+n}{2}$ contributors. And at the end of y years, there will be n widows; but supposing only one widow, then we must reckon $\frac{m+n}{2n}$ contributors, or which is the same thing, one contributor and $\frac{m+n}{2n}$

$\times$ an-

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annuity paid; put $\frac{m+n}{2n} = s$. Then will $x =$

$$\frac{+sv}{pt}$$

1. Let the time be 4 years, the middle between and 7; then $m=48$, $y=4$, $n=44$, $a+y=44$, $=2$, $v=3.63$, $p=12.85$, $t=.855$, $s=1\frac{1}{2}$.

Then $x = \frac{5.795}{10.98} = .52$, the widow's annuity from to 7 years.

2. The middle between 7 and 15 is $11=y$, then $+y=51$, $n=37$, $v=8.76$, $p=11.55$, $s=1.16$, $=.649$; then $x = \frac{12.15}{7.50} = 1.60$, the annuity from 7 to 15.

3. The middle between 15 and 48 is $32=y$, and $+y=72$, $n=16$, $v=17.87$, $p=6.07$, $t=.285$, $=2$. Then $x = \frac{37.74}{1.73} = 21.81$, the annuity from 15 to the end.

This Prob. may also be solved by Prob. IV. But this scheme is very imperfect. For the annuity should be computed for every year, or at least for every 5 years; and that for all ages between 21 and 45.

PROB. VII.

A person enters into society, and pays a fine f , and pounds a year for y years; after which he is to receive an annuity for life. But if he dies before y years he out, the payment ceases. To find this annuity for life; at 4 per cent. (Provident Society.)

This Prob. is resolved like the rest. Let $a =$ the age, m its complement. Then if there be m people at first, y of them will die in y years. And the money paid by these, excluding the fine, is $=yA$,

K 2

where

where A is the value of a life whose complement is y . This is at 1*l.* a year.

If $n = \text{comp. } a + y$; then these n contributors (in y years) pay nv , where v is the present value of 1*l.* annuity for y years.

After y years, there are also n annuitants. And they in all receive np , where p is the value of a life whose comp. is n . But the value of this, y years sooner, is tp , where t is the present worth of 1*l.* due y years hence.

But to reduce all to one person's annuity = divide by n , and then the money paid is $= \frac{m}{n} f +$

$\frac{byA}{n} + bv$; and the money received $= tp$.

Therefore put $m = \text{comp. of } a$,

$n = \text{comp. } a + y$.

$A = \text{number in Table III. against } 88 - y$

$v = \text{number in Table II. against } y$.

$p = \text{number in Table III. against } a + y$.

$t = \text{number in Table I. against } y$.

Then the annuity $x = \frac{\frac{m}{n} f + \frac{y}{n} A + v \times b}{pt}$.

E X A M.

Suppose the age $40 = a$, and $y = 10$, $f = 0$, $b = 12.8$; then $m = 48$, $n = 38$, $A = 3.92$, $v = 8.14$, $p = 11.75$, $t = .6755$; then $x = \frac{117.017}{7.93} = 14.75$

the annuity.

If $f = 31\frac{1}{2}$, $b = 8\frac{1}{2}$, then $x = 14.69$, the annuity.

Cor. If the conditions be to pay f fine, and b pounds a year, for half the complement of life (to 88); then

$y = \frac{1}{2} m$, and $A = p$. Whence $x = \frac{2f + A + vb}{At}$

And this is a very rational scheme to proceed on.

P R O B.

PROB. VIII.

A man whose age is a , enters into society, and pays 1. a year for life. What sum of money shall his heir &c. receive at his death; reckoning 4 per cent.

(Amicable Society.)

Let $n = 88 - a$, seek a in Table III. and put $s =$ its value. And suppose there are n persons at first, when 1 will die every year; and the amount of 1. present money for 1, 2, 3, &c. to n years, will be

$R + R^1 + R^2$ &c. to R^n , whose sum is $\frac{R^n - 1}{r} R$

for n persons, and the mean of all these sums is $\frac{R^n - 1}{nr} R$, or putting $M =$ amount of 1. for n

years, by Table I. then the mean $= \frac{M - 1}{nr} R$ the

sum that his heir, &c. shall have at his death, for 1. paid at first. And consequently $\frac{M - 1}{nr} RS$ is

the sum that his heir receives at his death for paying 1. annuity for his life, or S at first.

EXAM.

Suppose a life of 32, its value $S = 14.68$, at 1. a year for life; then $n = 56$, and $M = 8.992$, then

$\frac{M - 1}{nr} R = 3.71$; and $3.71 \times 14.68 = 54.46 =$

sum paid at his death, which is too much,

Or thus.

Suppose there be n persons, and each pays 1. at his death; then the present worth of 1. due 1, 2,

K 3

3, &c.

3, &c. years hence is $= \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3}$ &c. to

$\frac{1}{R^n}$ whose sum $= \frac{1 - \frac{1}{R^n}}{r} = \frac{1-p}{r}$; and the mean

present worth of 1*l.* for each person, is $= \frac{1-p}{nr}$,

where p = present worth of 1*l.* due n years hence.

But $\frac{1-p}{nr} : 1*l.* :: 1*l.* : $\frac{nr}{1-p}$ = money paid at his$

death, supposing 1*l.* paid at first. Or $\frac{nr}{1-p}$ S = money paid at his death, supposing S (or 1*l.* annuity) paid at first.

In a life of 32, $\frac{nr}{1-p} = 2*l.*52$ the money paid at last, for 1*l.* paid at first.

PROB. IX.

The same things supposed as in Prob. V. and that all the members are of the same age at first; and also that the same number of members is constantly kept up, by admitting new ones, when the old ones die. To find the number of contributors, and number of annuitants in any one year.

In the Schol. Prob. V. I shewed how to find the number of contributors and annuitants in any year, when no new members were admitted; in which case, one will die every year out of n (or 48), and therefore all will be dead in 48 years, the complement of life. But here as there are more members coming in, they will die faster and faster, till at last, or at the end of n (or 48) years, there will be people of all ages from 40, the year of admission, to the end of life; at which time, two will continually die every year, which I thus prove.

1. Sup-

1. Suppose there are n (or 48) couple, of the age of 40, 47 of the age of 41, 46 of the age of 42, 45 of the age of 43; and so on to 1. Then it is plain, by the laws of mortality, that one will die out of every parcel, every year; and therefore 48 will die in a year, out of the whole number; that is, out of $1+2+3+4$ &c. to 48, or 24×48 ; and consequently 1 out of 24 will die in a year, or 2 out of 48. And therefore one woman will die every half year. And always at the death of a woman a new couple enters.

2. To find the number of contributors yearly. By Cor. Prop. IV. annuities, the expectation of two equal joint lives is $\frac{1}{2}n$; and this must be supposed to be the life of a contributor. For in a great number, if some be longer, others will be shorter; and after $\frac{1}{2}n$ years, they will go off as fast as they come on; and after that will admit of no increase or diminution; therefore there will be then constantly $2 \times \frac{1}{2}n$, or $\frac{2}{3}n$ contributors.

3. For the number of widows. By Cor. 1, Prop. VI. annuities, the expectation of the longest of two lives is $\frac{2}{3}n$; therefore $\frac{2}{3}n$ is the life of the longest liver, be it man or woman, and therefore $\frac{1}{3}n$ is the length of the life of the survivor, after the death of the other; consequently in these $\frac{1}{3}n$ years $\frac{2}{3}n$ survivors will come in, as before of widows, and constantly be the same afterwards. But as half of this number is the number of surviving men, the other half or $\frac{1}{3}n$ will be the number of surviving women; that is, there will be constantly $\frac{1}{3}n$ widows or annuitants. All this will be more clear, by considering, that there are constantly n women, wives and widows; and so many wives so many contributors; and $\frac{2}{3}n$ contributors $\neq \frac{1}{3}n$ annuitants $= n$, the whole number.

4. To make an estimate for the intermediate years. Let d denote any year from the beginning;

K 4

then

then by a like computation I find, that after d years, 1 out of $n - \frac{1}{2}d$ persons, will die in a year, of the supplemental people; or that $\frac{d}{n - \frac{1}{2}d}$ persons will die in a year.

In the series A below, one person dies every year, which is to be supplied by a new one coming in. But instead of one, let 48 come in, (that 1 may die in a year), as is denoted by the series B, C, D, &c. out of each of which also 1 dies in a year.

A = 48, 47, 46, 45, 44, 43, 42, &c.

B = 48, 47, 46, 45, 44,

C = 48, 47, 46, 45,

D = 48, 47, 46,

&c. Now proceeding as in Schol. Prop. V. assume any number for d , the year. The number for d

in the series A is $\frac{ll}{n}$. For $d-1$, in the series B, is

$\frac{l+1^2}{n}$. For $d-2$, in series C is $\frac{l+2^2}{n}$. For

$d-3$, in series D, is $\frac{l+3^2}{n}$ &c. for $d-1$ terms

beginning at B. The sum of all which (divided by n) is the number of contributors; that is,

$\frac{l+1^2}{nn} + \frac{l+2^2}{nn} + \frac{l+3^2}{nn}$ &c. = number of con-

tributors, the sum of all which (by the differential method) is = $\frac{2nn-3n+1}{6nn} n - \frac{2ll+3l+1}{6nn} l =$

(putting $n-d$ for l) $\frac{n-d \times d-1}{n} + \frac{2dd-3d+1}{6nn}$

$d = \frac{n-d \times d-1}{n} + \frac{2d-3}{6nn} dd$ nearly.

5. In like manner for the widows, the number of d in the series A, is $\frac{dl}{n}$. For $d-1$, in series

is $\frac{\overline{d-1} \cdot \overline{l+1}}{n}$. For $d-2$, in series C, is

$\frac{\overline{d-2} \cdot \overline{l+2}}{n}$; for $d-3$, in series D, is $\frac{\overline{d-3} \cdot \overline{l+3}}{n}$

&c. for $d-1$ terms, beginning at B. The sum of which (by the differential method) divided by n , is

$\times \overline{d-1} + \frac{\overline{d-l} \cdot \overline{d-1}}{2nn} d - \frac{2dd-3d+1}{6nn} =$ (put-

ting $n-d$ for l) $\frac{\overline{d-1}}{2n} d - \frac{2d-3}{6nn} dd$, for the

number of widows.

6. But as no supply has been made for the people dying in B, C, D, &c. which in n years are reduced to half. Therefore multiply each quantity by

$+\frac{d}{n}$, and putting $P = \frac{2d-3}{6nn} dd$, then we have

$+\frac{d}{n} \times : \frac{\overline{n-d} \cdot \overline{d-1}}{n} + P :$ for the number of

contributors. And $\frac{n+d}{n} \times : \frac{\overline{d-1}}{2n} d - P :$ for

the number of annuitants; exclusive of the series

A. And the same in proportion for any number greater than n .

7. Therefore if N be the number of couples in this society. The number of contributors at the

end of d years, will be $\frac{ll}{nn} N + \frac{n+d}{nn} N \times :$

$\frac{\overline{d} \cdot \overline{d-1}}{n} + P :$

And

And the number of annuitants upon the life will be $\frac{dl}{nn} N + \frac{n+d}{nn} N \times \frac{d-1}{2n} d - P$: nearly, including the original number (A).

8. And after n years, the constant number of contributors, will be $\frac{2}{3} N$. And the constant number of annuitants, $\frac{1}{3} N$, nearly.

A R T. IV.

O F T H E

MOON'S MOTION.

P R O B. I.

To find the absolute Force whereby the Moon is drawn from its Orbit, by the Action of the Sun ; supposing the Moon's Orbit nearly circular.

ET QAZ be the moon's orbit, P the place of the moon at any time. Let the sun's distance be r , moon's distance $PT = a$, sine $PK = y$, arch $QP = z$, versed sine $QK = x$, p = earth's periodical time in seconds = 31556940, g = force of gravity, b = space descended in a second by gravity, $\pi = 3.14159265$.

The arch described by the earth in a second is $\frac{\pi r}{p}$, and $\frac{4\pi\pi r r}{2rpp}$ or $\frac{2\pi\pi r}{pp}$ is its versed sine, and this is the space through which the earth is drawn by the sun in a second, which measures the sun's force. By Cor. 1, Prop. 27, centr. forces, the sun's disturbing force at P = $\frac{2y}{r} C = \frac{2y}{r} \times \frac{2\pi\pi r}{pp}$

= $\frac{4\pi\pi y}{pp}$; and this is the difference of the forces

at K and P; and therefore $\frac{4\pi\pi y}{pp}$ is the force that draws the moon from her orbit, directly towards the sun, or rather the space through which it is drawn from its orbit in a second. Hence, if F be the

Fig. the force required, then $b : g :: \frac{4\pi\pi y}{pp} : F = \frac{4\pi\pi y}{pp}$

$$= \frac{y}{b} \times \frac{g}{25224933618000}$$

Cor. If s be the sine of QTP to the radius 1, the

$$y = sa, \text{ and } F = \frac{4\pi\pi sa}{ppb} g = \frac{sg}{21432.5b}$$

P R O B. II.

- I. To find how far the moon is drawn from its orbit in any time, from the quadrature, by the sun's disturbing force, supposing her orbit nearly circular.

Let QPZ be her original orbit, P her place. Suppose a body P revolving in the orbit QPZ, as the moon would do if it was not disturbed by the sun. And suppose the line PS always drawn from the body P to the sun; and that the moon is drawn away in the line of vagation PS to D, having described P D in the time of moving from Q to P. To find the length P D; and the velocity at D, along the line PS.

Denoting the quantities as before, also putting t = the moon's synodic time in seconds, P D =

Then $\frac{2\pi a}{t}$ = arch of the moon's orbit described

in a second. And $2\pi a : t :: z : \frac{tz}{2\pi a}$ = time of

passing through z . But (by Prob. I, Section III

Fluxions) $\dot{v} \propto F$ (v being the velocity at D)

and in the case of falling bodies, $\dot{v} = 2bt$, and

$F = b$. Therefore $2bt : bt :: \dot{v} : \frac{4\pi\pi y}{pp} \times \frac{tz}{2\pi a}$

IV. THE MOON'S MOTION.

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$$1 : 1 :: \dot{v} : \frac{2\pi y \dot{z}}{app}, \text{ therefore } \dot{v} = \frac{4\pi t}{app} \times y \dot{z} =$$

$\times a\dot{x}$, by nature of the circle. Therefore $v =$

$$\frac{x}{31059503}, \text{ or } v = \frac{a}{31059500} \times \text{versed}$$

of the moon's distance from the last quadrature, radius = 1. And supposing $V =$ that versed sine, $a = 1176948000$; then $v = 37.8934V$.

Again, (Fluxions, *ib.*) $s \propto vt$, and in falling bodies $s = 2bt$, and $v = 2b$; therefore $2bt : 2b$

$$:: \dot{w} : \frac{4\pi t x}{pp} \times \frac{t \dot{z}}{2\pi a}. \text{ Therefore } \dot{w} = \frac{2tt x \dot{z}}{app}$$

$$\frac{2tt x}{app} \times \frac{a\dot{x}}{y} = \frac{2tt}{pp} \times \frac{x\dot{x}}{y} = \frac{2tt}{pp} \times \frac{x\dot{x}}{\sqrt{2ax - xx}}$$

$$\frac{4tt}{ppa} \times \frac{ax\dot{x}}{2\sqrt{2ax - xx}}. \text{ But the fluent of}$$

$$\frac{ax\dot{x}}{\sqrt{2ax - xx}} \text{ is known to be } = \text{segment QHP.}$$

$$\text{Therefore } w = \frac{4tt}{ppa} \times \text{segment QHP. Or } w =$$

$$\frac{4ta}{pp} \times \text{segment QHP, to the radius } 1 = \frac{a}{38.2436255}$$

segment QHP, radius = 1.

Cor. 1. Hence in the syziges at A, the moon is drawn from its orbit, with a velocity of 37.894 feet a second. And at the next quadrature with a velocity of 75.788 feet, where the force nearly ceases.

Cor. 2. And in the syziges she has departed 9402985 feet, or 1780.87 miles from her orbit. And at the next quadrature 51752450 feet, or 9801.6 miles, which is about $1\frac{1}{4}$ the earth's diameter, or $4\frac{1}{2}$ of the moon's, amounting to about $4^\circ 42'$.

Cor.

Fig. Cor. 3. *The moon's perp. distance from its true orbit in any place, is as $w \times \text{fine of the arch } QP$.*

For the perp. distance of Δ from the orbit PQ is $P\Delta \times \text{fine of the angle } \Delta PA \text{ or } QTP$.

S C H O L.

1. Strictly speaking, the moon perhaps may not always keep in the line PS , but fall a little back; or be a little forward, in different parts of the orbit. But this cannot be certainly known; for want of knowing the original orbit.
2. Besides the line $P\Delta$ described by the moon, has been here supposed to be a right line, upon supposition, that the sun stands still, and has no motion in longitude. But since in the time of half a synodic revolution, the sun has moved forward about $14\frac{1}{2}$ degrees, therefore the line described by the moon, moving towards the sun, will be in the form PBM ; of such a nature that the extreme tangents $P\Delta$, MC will intersect at D in an angle of $14\frac{1}{2}^\circ$. And the radius of curvature at any point B , will be as the velocity of the moon at B , that is, as the versed sine of the correspondent arch QH . Supposing PB described in the time QH is described by P .

After a like manner the velocity and space described, along the curve QP , or along the radius TP may be calculated. And since I have performed the calculations, by a method somewhat different, I shall here set down the result thereof.

The velocity gained,

along the curve at A	—	20.32 feet.
in the radius at A	—	32 04
in the line PS at A	—	40.64

The

The space gained,

in the curve	at A	—	$\frac{1}{194}$	of TP.
in the radius TP	at A	—	$\frac{1}{168}$	of TP.
in the line PS	at A	—	$\frac{1}{134}$	of TP.

P R O B. III.

To find the place where the moon is furthest from the original orbit, in the line perpendicular to it.

3.

Let O be the moon's place in the time of vagation KPO, draw TP from the center T, and parallel to the orbit at P. Let QK = x, KP = y, PO = w, QT = 1, then TK = x - 1. When 1 : y :: w : P = wy = maximum, and yw

wy = 0, that is, $\frac{2ty}{pp} \times x\dot{x} + \frac{4ty\dot{y}}{ppa} \times \text{feg. QHP}$

= 0, and $x\dot{x} + \frac{2y\dot{y}}{a} \times \text{feg. QHP} = 0$. Or $x\dot{x} +$

$\frac{x\dot{x} - 2x\dot{x}}{ay} \times \text{feg. QHP} = 0$, or $ayx + \overline{2a - 2x} \times$

$\text{feg. QHP} = 0$, or $\frac{1}{2}ax = \frac{x-a}{y} \times \text{QHP}$; put

= 1, therefore $\frac{\frac{1}{2}xy}{x-1} = \text{feg. QHP} = \frac{1}{2} \text{ arch}$

$\text{QHP} - \frac{1}{2}y$. And $\text{arch QHP} = y + \frac{xy}{x-1} =$

putting TK = c) $y + \frac{1+c.y}{c} = 2y + \frac{y}{c} = \tan.$

ZP + 2 sine of ZP. Whence arch QP = 135°

6' nearly. And then P.D = $\frac{a}{59}$ the greatest dis-

tance from her orbit, and Po = $\frac{a}{41}$.

Or

Or thus.

4. Draw Δ ATD and Δ PKG, and put $x = \Delta A$ then rect. $A \Delta D = \text{rect. } P \Delta G$, or $xx + 2ax = w^2 + 2yw$, and x is to be a maximum, put m for x then $m^2 + 2am = w^2 + 2yw$, in fluxions $2mw + 2a\dot{m} = 2w\dot{w} + 2y\dot{w}$; that is, $\overline{w+y} \times \frac{x\dot{x}}{y} + \frac{2\text{seg. QHP}}{a} \times \frac{a\dot{x} - x\dot{x}}{y} = 0$, and $wx + yx + \frac{2\text{ seg. QHP}}{a} \times \frac{a\dot{x} - x\dot{x}}{y} = 0$, and $\frac{\frac{1}{2}a\dot{y}x}{x-a} = \text{seg. QHP} - \frac{wx}{2x-a}$
 $= \text{seg. QHP} - \frac{2ax}{pp. x-a} \times \text{seg. QHP}$; where, the last term be neglected, as being very small, it comes to the same as the first.

Put $a = 1$, $x - 1 = c$, then $\frac{\frac{1}{2}axy}{x-a} = \frac{1}{2} \times \frac{1+c}{c}$

$$y = \frac{1}{2} \frac{y}{c} + \frac{1}{2} y = \frac{\tan. ZP + S. ZP}{2} = \text{seg. QHP} -$$

$$\frac{1+c}{c} \times \frac{\text{seg. QHP}}{76.4872}. \text{ For seg. QHP, put arch}$$

$$\frac{QP - PK}{2} = \frac{QP - y}{2}, \text{ and prosecuting the calcu-}$$

lation, there will be found x or $\Delta A = .0153 = \frac{a}{65.5}$, and $ZP = 43^\circ 33'$, or $QP = 136^\circ 27'$, PA

$$= 1^\circ 6', P \Delta \text{ or } w = .02213 = \frac{a}{45}$$

P R O B

PROB. IV.

To find the area QP D, described by the line of variation P D.

The fluxion of the area QP D is $P D \times \dot{x} = w\dot{x}$

$\frac{4tt\dot{x}}{ppa} \times \text{segment QHP}$. But segment QHP =

$\frac{ya}{2}$; therefore the flux. area = $\frac{4tt\dot{x}}{ppa} \times \frac{za - ya}{2}$

$\frac{2tt}{pp} \times \overline{zx - yx}$. But fluent of $y\dot{x} = \text{area QKP}$,

and let the fl. $z\dot{x} = zx + s$, then $z\dot{x} = z\dot{x} + x\dot{z} + s$,

and $s = -x\dot{z} = -\frac{ax\dot{x}}{y} = -\frac{ax\dot{x}}{\sqrt{2ax - xx}}$, and

therefore s or the fl. $\frac{ax\dot{x}}{\sqrt{2ax - xx}} = 2 \text{ area QHP}$.

therefore the area QP D = $\frac{2tt}{pp} \times : zx + 2 \text{ area}$

HP - area QKPH : = $\frac{2tt}{pp} \times : zx + \text{seg. QHP}$

triangle QKP : = $\frac{2tt}{pp} \times : zx + \frac{\overline{z - y} \cdot a - yx}{2}$

here $\frac{2tt}{pp} = \frac{1}{76.48725}$.

Cor. 1. In the syziges at A, the area will be =

$aa \times 3.1416 - aa : \times \frac{2tt}{pp} = \frac{aa}{56.4}$.

Cor. 2. At the next quadrature Z, the area =

$\frac{2tt}{pp} \times 2 \frac{1}{2} aa \times 3.1416 = \frac{aa}{9.74}$.

L

PROB.

THE MOON'S MOTION.

PROB. V.

To find nearly the original orbit of the moon.

1. We found in Prop. II. how far the moon is drawn from its orbit, at any place therein, or in describing any arch. But since the time of describing the arch is proportional to the synodic revolution; the arch must be reduced to the periodical time. Therefore in the quadrant $QA = 90^\circ$, say as the synodic, to the periodic time; so is 90° , to $83^\circ 16'$. The area of the segment QHP for $83^\circ 16'$ is 23011; whence the line of vagation PD is

$$\frac{1}{166.18} a.$$

5. Let QAZ be the present orbit. Make $QA = 90^\circ$, and draw AT, and set $\frac{1}{166} TQ$ from A to a, and the semi-circle QaN being drawn will be nearly the original orbit, or that described without perturbation, wherein $ZN = \frac{1}{33} TZ$. And $TZ : TN :: 83 : 82$. The same contraction will be made in the other half of the orbit. And therefore the diameter of the present orbit, is to the diameter of the original orbit, as 83 to 82.

If the arch of 90 had not been reduced to $83^\circ 16'$ the diameter of the disturbed orbit, would be to the diameter of the undisturbed orbit, as 67 to 66.

Cor. Hence, by the action of the sun, the moon's orbit is lengthened at least $\frac{1}{33}$ part of itself; and the periodical time is increased accordingly; which will be $\frac{1}{33}$ part, or near 12 hours more.

SCHOL.

From the foregoing problems, it appears how much the moon's motion is disturbed, and her orbit distorted, merely by the sun's variable force acting in

ART. IV. THE MOON'S MOTION. 147

ing upon her. From which so many irregularities Fig. arise, both in her place and distance; that no tables have ever been made, to answer truly for any considerable length of time.

If her undisturbed orbit and periodical time in were given; and the time of the moon's being in the quadrature known; I doubt not but one might find the moon's place very near, by the second Prob. for a given time; and be able to compose tables to find her place from any time till the next quadrature, but no longer. For after that she will be put into a new tract, which being disturbed over again, will differ from the former, and after that into another still different, and so on. So that after every semi-revolution she will be put into a new path different in some degree from the former. Yet in time she may describe orbits which will converge to some regularity. But I have neither time nor leisure to prosecute these things any further.

The action of the sun on the further side of the moon's orbit is something less than that on the nearest side, which also causes some error. But then the moon (as all the other satellites do) moving the same way about the earth, as the earth moves about the sun; the centrifugal force of the moon is greater on the outside, which compensates for the less force of the sun. For on the outside of the orbit, the moon's absolute motion is the sum of the motions of the earth and moon, in their orbits; but on the inside it is their difference; and the centrifugal forces are as these motions.

A R T. V.

The Construction of the Arches of Bridges, &c.

UPON account of the great use, importance, and curiosity of this subject, I took the pains, in my book of *Fluxions*, to calculate the nature of the curve, upon which all the parts of an arch will be in equilibrio. And in the *Mechanics*, I have given numbers for the construction of such arches. For no architect or engineer that knows what he is about, will venture to build any arches, but what shall be the strongest and most durable that can be made. And that can only be when all the parts are in an exact equilibrio; so that they cannot thrust out one another, and fall down. And here it is supposed the stones have no cohesion with one another, but are entirely sustained by the figure of the arch. So that if they were laid in oil, and could easily slide by one another, they would still keep their places. But being laid in lime or mortar to make them cohere, they could then bear any additional weight, that could come upon them. I shall, in what follows, give the investigation of the numbers by which the curve is to be drawn, together with some other things relating to the subject.

P R O P.

PROP. I. PROB.

Several inflexible right lines, moveable about the points B, F, H, L, O, being given in a vertical plane; but the extreme points A, R, fixt immoveable. To find the proportion of the weights or forces insisting in the angles B, F, H, &c. so that the figure may stand in equilibrio, in that position.

Through the angles B, F, H, &c. draw perpendiculars to the horizon, Bc, Fe, Hi, &c. which are the direction of the weights or forces; and taking Bc of any length compleat the parallelogram $aBbc$; likewise take $dF = Bb$, and compleat the parallelogram $dFfe$. In like manner make $gH = Ff$, $kL = HI$, $nO = Ll$, compleating all the parallelograms Hbi , $kLlm$, $nOop$. Then the forces or weights on the angles B, F, H, L, O, are respectively as the diagonals Bc, Fe, Hi, Lm, Op.

For the force Bc is equivalent to Ba and Bb. Ba acts towards the immoveable point A, and Bb acts towards F. Also the force Fe is resolved into the forces Fd and Ff. But Fd acts towards B, and is opposed and kept in equilibrio by the equal force Bb, because $Bb = dF$, by construction. After the same manner the force Hi is resolved into Hg, Hb: the force Lm into Lk, Ll: the force Op into On, Oo. Of these the forces Ff, Hg; and Hb, Lk; and Ll, On, destroy one another, being equal and contrary; and the force Oo acts against the immoveable point R; and therefore all the points B, F, H, L, O, will remain unmoved.

After the same manner may the forces be found, when they act not perpendicular to the horizon, but tend towards a given point; and that is, by drawing the lines Bc, Fe, Hi, &c. to that point, and

Fig. completing the parallelograms, and proceeding 6. as before.

Now the force on any point B, is to the force on any other point O :: Bc to Op. But Bc to Op is in the complicate ratio of

$$\left\{ \begin{array}{l} Bc \text{ to } Bb \text{ or } dF \\ dF \text{ to } de \text{ or } gH \\ gH \text{ to } gi \text{ or } Lk \\ Lk \text{ to } km \text{ or } nO \\ nO \text{ to } Op \end{array} \right\} \text{ that is, } \left\{ \begin{array}{l} cab : cBa \\ eFH : eFB \\ iHL : iHF \\ mLO : mLH \\ pOR : ROL \end{array} \right\} \text{ as the } \left\{ \begin{array}{l} \text{fines of} \end{array} \right.$$

But $s.eFH$, and $s.iHF$ are ratio's of equality; and $s.iHL$ and $s.mLH$ are ratio's of equality. Therefore $Bc : Op :: s.cab \times s.mLO \times s.pOR : s.cBa \times s.eFB \times s.ROL :: s.ABF \times s.pOL \times s.pOR : s.cBA \times s.cBF \times s.ROL :: \frac{s.ABF}{s.cBA \times s.cBF}$

$$\frac{s.ROL}{s.pOL \times s.pOR} :: \text{force on B} : \text{force on O.}$$

Cor. 1. The forces on any two angles B, L, in directions BA, LO; are directly as the secants, or reciprocally as the cosines of elevations, that is, as $s.mLO$ to $s.cBA$.

For the forces Ba and Ll, are in the complicate ratio of

$$\left. \begin{array}{l} Ba \text{ to } Bc \\ Bc \text{ to } Lm \\ Lm \text{ to } Ll \end{array} \right\} :: \frac{s.cBF : s.ABF.}{\frac{s.ABF}{s.cBA \times s.cBF} : \frac{s.HLO}{s.mLH \times s.mLO}} \left. \begin{array}{l} \\ \\ \end{array} \right\} \frac{s.HLO}{s.mLH.}$$

$$:: \frac{1}{s.cBA} : \frac{1}{s.mLO}$$

Cor. 2. The weights or forces upon the angles will continue the same; if the inclinations of the lines to one another, and to the horizon, continue the same; though the lengths of the lines be never so much varied.

For the lengths of the lines do not come into computation. Fig. 6.

Cor. 3. The direction of the force on any point F is the line FB downwards, and in the line FH upwards.

This appears by the construction.

Cor. 4. The force insisting on any angle F is as the sum of the cotangents of eFB , eFH , when they are both acute; but as the difference, when one is obtuse.

For the force on F is as $\frac{s.BFH}{s.eFB \times s.eFH}$. But

$BFe = s.BFe \times \cos. eFH + s.eFH \times \cos. BFe$.

Therefore the force on F is as

$$\frac{s.BFe \times \cos. eFH + s.eFH \times \cos. BFe}{s.eFB \times s.eFH} = \frac{\cos. eFH}{s.eFH}$$

$$\frac{\cos. BFe}{s.BFe} = \cot. eFH + \cot. BFe. \text{ But here the}$$

angle eFH being obtuse, the cosine and cotan. will be negative; therefore the force at F is as $\cotan. BFe - \cotan. eFH$.

Cor. 5. If any external angle ABx be infinitely small, then any weight pressing at B will act as much against F as against A.

For if ABx or Bac be infinitely small, ac coincides with aB ; and the forces in directions BF , BA (being as the equal lines ac , aB) will be equal.

PROP. II.

If a weight is sustained by the curve AFV , in a vertical plain; and is kept in equilibrio by the weight insisting on every point. Then the weight or pressure upon any point F is as the curvature at F directly, and the square of the sine of the angle eFy or ITF reciprocally: TFy being a tangent, and TI perp. to the horizon.

For draw the external angle BFy by producing HF (Fig. 6.) and suppose all the lines AB , BF , FH ,

Fig. FH, &c. to be equal. Then (by Prop. I.)

7. force or pressure on any angle F, is as $\frac{s.BFH}{s.BFe \times s.eFH}$

Now let the number of the equal lines be augmented, and their length diminished ad infinitum, and the figure AFOR will become a curve line, and the pressure will act on every point of the curve; and the infinitely small evanescent angle BFy becomes the angle of contact, and $\angle FFB$ becomes $= eFy$; and the pressure at F being

$$\frac{s.BFH}{s.BFe \times s.eFH} \text{ or } \frac{s.BFy}{s.eFB \times s.eFy} \text{ is } = \frac{s.BFy}{s.eFy^2}$$

when the infinitely small part of the curve is given, the sine of the angle of contact BFy is as the curvature. Therefore (in Fig. 7.) the force on F is as the curvature at F directly, and the square of the sine of eFy or ITF reciprocally.

8. Cor. 1. If a weight like a wall APQR be incumbent on the arch AFR, standing in a vertical plane, and all the parts kept in equilibrio; then I say the height Fg on any point F, is as the curvature at F directly, and the cube of the sine of ITF reciprocally.

For the weight on the given part of the curve

Fr is as $\frac{C}{s.ITF^2}$, C being the curvature at F, and

the weight of the column Fgbr is as $Fg \times wr = Fg \times Fr \times s.FTI = Fg \times s.FTI$, because Fr is given.

Therefore $Fg \times s.FTI$ is as $\frac{C}{s.FTI^2}$, and Fg as

$$\frac{C}{s.FTI^3}, \text{ to keep the parts in equilibrio.}$$

Cor. 2. If R be the radius of curvature in F, then the height Fg is as $\frac{1}{R \times s.FTI^3}$.

Cor.

Cor. 3. If $ZI = x$, $FI = y$, $ZF = z$; then, Fg Fig. 8.
as $\frac{-x\ddot{y}}{y^3}$, where $\dot{x}$ is constant.

For $\dot{z} : y :: FT : TI :: \text{rad. } 1 : s.FTI = \frac{\dot{y}}{z}$,

therefore $Fg = \frac{\dot{z}^3}{R \times y^3}$. And R may be found by supposing either $\dot{x}$ or $\dot{y}$ given. If $\dot{x}$ be given, $R = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$. Therefore Fg is as $\frac{\dot{z}^3}{y^3} \times \frac{-x\ddot{y}}{\dot{z}^3} = \frac{-x\ddot{y}}{y^3}$.

In my book of Fluxions I have calculated the height Fg , for several curves by the method of Fluxions, by Cor. 3. I shall in the next Prop. do it by Cor. 2. for the sake of such as do not understand Fluxions.

PROP. III. PROB.

The nature of the curve $AFZR$ being given for the 8.
figure of an arch; to find the height Fg , of the wall
consisting thereon, at every point F ; so that all the parts
shall remain in equilibrio.

Draw the ordinate FI , and let $ZI = x$, $FI = y$,
 $ZF = z$, $ZS = a$, $R = \text{rad. curvature in } F$, draw
the tangent FT , which will be found from the na-
ture of the curve. Find the $S.$ angle FTI ; then
take Fg as $\frac{1}{R \times s.FTI^3}$, for the height.

EXAM. I.

Let FZ be the arch of a circle, radius $= r$,
 $ZI = x$, cos. $FZ = c$, $ZS = a$, then $c = s.FTI$, whence
 Fg is as $\frac{1}{rc^3}$ or as $\frac{1}{c^3}$, which at Z is $\frac{1}{r^3}$. Therefore

$$\frac{1}{r^3} : \frac{1}{c^3} :: a : Fg = \frac{ar^3}{c^3}.$$

And

Fig. And subtracting ZI or $x = r - c$, then $ZS = \frac{ar^3}{c^3} - \overline{r - c}$, for the height in any place F , above Z

E X A M. 2.

9. Let the curve FZ be an ellipsis, $t = AC$ the semi-transverse parallel to the horizon, $c = ZC$ the semi-conjugate, $ZI = x$, $FI = y$, $ZS = a$, $\pi = FQ$ perp. to the tangent FT . Then (Ellipsis, Prop. XI. Cor.

$$1.) \quad cc : tt :: \overline{c - x} (CI) : \frac{tt}{cc} \times \overline{c - x} = QI. \text{ And}$$

$$FQ (\pi) : \text{rad. } (1) :: QI : S.QFI \text{ or } FTI = \frac{tt \times \overline{c - x}}{cc\pi}. \text{ And rad: curvature at } F = \frac{4\pi^3}{bb},$$

$$\text{being the parameter (to } CZ) = \frac{2tt}{c}. \text{ Therefore}$$

$$\frac{I}{R \times S.FTI^3} = \frac{bb}{4\pi^3} \times \frac{c^6\pi^3}{t^6 \times \overline{c - x}^3} = \frac{4t^4}{4cc\pi^3} \times$$

$$\frac{c^6\pi^3}{t^6 \times \overline{c - x}^3} = \frac{c^4}{tt \times \overline{c - x}^3}, \text{ which is as } Fg; \text{ whence}$$

$$QZ = \frac{tt}{c}, \text{ and } ZS \left(\frac{c^4}{ttc^3} \right) : a :: \frac{c^4}{tt \times \overline{c - x}^3} : Fg$$

$$= \frac{c^3a}{c - x^3}. \text{ Also } \frac{c^3a}{c - x^3} - x = \text{the height of } g \text{ above } Z.$$

E X A M. 3.

10. Let AFZ be another ellipsis, whose semi-transverse CZ is perp. to the horizon, and semi-conjugate AC equal to the semi-transverse of the other. Let $AC = t$, $ZC = n$, $ZI = x$, $FI = y$, and equal to the former FI . $ZS = a$, $\pi = FQ$ perpen. to the tangent FT . Then as before, the parameter $b =$

$$\frac{zt}{n}, \text{ and } QI = \frac{tt}{nn} \times \overline{n - x}, \text{ and } S.FTI =$$

and the radius of curvature at F = 10.

$$= \frac{t^4}{nn^3}; \text{ therefore } \frac{I}{R \times S.FT^3} = \frac{n^4}{tt \times n-x^3}$$

$$\text{Fg, and } Fg = \frac{n^4 a}{n-x^3}.$$

And $\frac{n^4 a}{n-x^3} - x = \text{height of } g \text{ above } Z.$

SCHOL.

From these three examples it will appear what is comparative strength of these three arches, the ellipsis, the circle, and the sharp ellipsis, having all of them the same breadth, and the same height ZS at the top Z. For taking FI the same wall; then in the flat ellipsis (Fig. 9.) $tt : cc ::$

$$cc - zz, \text{ and } zz = cc - \frac{ccyy}{tt} \text{ and } \frac{zz}{cc} = 1 -$$

$$\frac{tt - yy}{tt}, \text{ and } \frac{cc}{zz} = \frac{CZ^2}{CI^2} = \frac{tt - yy}{tt}, \text{ or } Fg =$$

$$\frac{Z^2}{CI^2} = \frac{tt - yy}{t^3} \sqrt{tt - yy}, \text{ where } t \text{ and } y \text{ being al-}$$

ways the same, the height Fg is precisely the same, in all the three figures.

But if the top of the wall is bounded by a horizontal right line passing through S, the height then will be $Fg - a - x = b - x$. Therefore when x is greatest, the height above S is the least, which is in the sharp ellipsis, and in this figure the want of balance at F is the least; and in the flat ellipsis, it is the greatest. Therefore upon that account, the sharp ellipsis is strongest, the circle next, and the flat ellipsis weakest.

E X A M.

Fig.

E X A M. 4.

8. Let FZ be a parabola, $rx = yy$, the subtangent $IT = 2x$, and $FT = \sqrt{yy + 4xx} = \sqrt{rx + 4xx}$
 $FT : FI :: \text{rad. (1)} : \text{S.FTI}$, that is, $\sqrt{rx + 4xx}$
 $\sqrt{rx} :: 1 : \sqrt{\frac{rx}{rx + 4xx}} = \text{S.FTI}$, and rad. curvature at F = $\frac{rr + 4rx^{\frac{3}{2}}}{2rr}$, whence $\text{Fg} \left(\frac{1}{R \times \text{S.FTI}^3} \right)$
 is as $\frac{2rr}{rr + 4rx^{\frac{3}{2}}} \times \frac{rx + 4xx^{\frac{3}{2}}}{rx^{\frac{3}{2}}} = \frac{2rr}{r^3}$ a given quantity. And since $ZS = a$, the height every where is equal to a .

I omit the hyperbola, because the curve PSQ continually approaches to the hyperbola, and therefore is unfit for an arch.

E X A M. 5.

Let FZ be the catenary. Then by the nature of the curve $zz = 2rx + xx$, and subtangent $IT = \frac{2y}{r}$
 and $FT = \sqrt{yy + \frac{zzyy}{rr}}$, and $\sqrt{yy + \frac{zzyy}{rr}} : y :: \text{rad. } 1 : \text{S.FTI} = \frac{y}{\sqrt{yy + \frac{zzyy}{rr}}} = \frac{1}{\sqrt{1 + \frac{zz}{rr}}} = \frac{r}{\sqrt{rr + zz}}$; and the rad. curvature at F = $\frac{rr + zz}{r}$
 Whence $\text{Fg} \left(\frac{1}{R \times \text{S.FTI}^3} \right)$ is as $\frac{r}{rr + zz} \times \frac{rr + zz^{\frac{3}{2}}}{r^3} = \frac{\sqrt{rr + zz}}{rr}$; and in Z it is $\frac{1}{r}$, therefore

V. OF ARCHES.

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$$a :: \frac{\sqrt{rr+zz}}{rr} : Fg = \frac{a\sqrt{rr+zz}}{r} = \frac{a \times \overline{r+x}}{r} \text{ Fig. 8.}$$

$$+ \frac{ax}{r}.$$

Cor. 1. If $a = r$, then Fg is every where $= IS$, PSQ is a right line.

Cor. 2. If a be greater than r , Fg will be greater IS ; and therefore the curve PS runs upwards, is convex towards Z .

Cor. 3. If a be lesser than r , then Fg will be lesser IS , and the curve PS runs downwards, or is concave towards Z .

Moreover if a be very small, the perp. thickness of arch is the same every where.

For if p be the perp. thickness of the arch; then $p :: \text{rad.} : S.FT :: FT : FI$; that is, FT

$$\left(\sqrt{yy + \frac{zzyy}{rr}} \right) : FI (y) :: Fg \left(\frac{\overline{r+x} \times a}{r} \right) : p =$$

$$\frac{\overline{r+x}}{yy + \frac{zzyy}{rr}} = \frac{a \times \overline{r+x}}{\sqrt{rr+zz}} = \frac{a \times \overline{r+x}}{\sqrt{rr+2rx+xx}} =$$

$$\frac{\overline{r+x}}{\overline{r+x}} = a : \text{consequently a heavy flexible line}$$

into this figure would support itself.

E X A M. VI.

Let FZ be the logarithmic curve, HD the II.

asymptote, FT a tangent at F . $ZD = r$, subtangent $HR = t$, which is always the same. Then

$H = r + x$, $FR = \sqrt{tt + rr + 2rx + xx}$, then

$(\sqrt{tt + rr + 2rx + xx}) : HR (t) :: \text{rad.} (1) :$

HFR or $FTL = \frac{t}{\sqrt{tt+rr+xx}}$. And the rad. cur-

vature

Fig.

11. vature at F = $\frac{tt + FH^2|^{\frac{3}{2}}}{t \times FH} = \frac{tt + r + x^2|^{\frac{3}{2}}}{t \times r + x}$. The

fore Fg is as $\frac{t \times r + x}{tt + r + x^2|^{\frac{3}{2}}} \times \frac{tt + r + x^2|^{\frac{3}{2}}}{t^3} = \frac{r + x}{t^2}$

which at Z is $\frac{r}{tt}$, therefore $\frac{r}{tt} : a :: \frac{r + x}{tt} : Fg$

$\frac{a}{r} \times r + x$, the same as for the catenary.

Cor. Hence if $a = r$, then Sg becomes the asymptote DH, and is therefore a right line.

If a is less than r , the curve Sg is concave towards Z, descending towards g. If a be greater than r , it is convex towards Z, rising upwards.

PROP. IV.

12. If there be two arches of equilibration of the same kind, AFZR and AfzR, upon the same base AR, the strength of the high arch is to the strength of the flat arch throughout; reciprocally as the radius of curvature at z to that at Z.

In the Schol. to Exam. 3, of the last Proposition, I shewed the weakness of the same arch in different points, by an undue weight. Here I shew the comparative strength in different arches of the same kind, by which I mean those whose ordinates NF are every where in a given ratio.

The materials whereof any bridge is made, are not of infinite strength, but there is a certain degree of pressure that will crush them to pieces, and thrust out the buttment.

Let H (in Fig. 6.) be the vertex of the figure. In the demonstration of Prop. I. I have shewed that if Hg, Hb are the forces along the lines Hg, Hb

L, that the weight H_i is the diagonal of the parallelogram $gHbi$. Now suppose gH , Hb , the elements of a curve, and the arches gH , Hb , to be given, being the greatest forces they can sustain; then H_i is twice the versed sine of the arch gH , which represents the weight. Therefore $\frac{gH^2}{\frac{1}{2}H_i} =$

twice the radius of curvature $= 2R$, or $\frac{gH^2}{H_i} = R$,

hence R is reciprocally as the weight H_i , or the weight H reciprocally as the radius of curvature H . Therefore (in Fig. 12.) the weight at Z to the weight at z , is as rad. curvature at z , to rad. curvature at Z ; which is all these bridges can sustain.

Cor. 1. If AFZ , Afz be two ellipses; their strengths are as their heights CZ and Cz . And so it is for the circle and ellipsis.

For the radius of curvature at Z is $\frac{AC^2}{CZ}$, and

the rad. curvature at z is $\frac{AC^2}{Cz}$; and these are as CZ to Cz .

Cor. 2. If $AfzR$ be an ellipsis, the strength of the arch at the vertex z , is no greater than that of a circle, whose radius is $\frac{AC^2}{zC}$, the radius of curvature in z .

PROP. V.

In an arch of equilibration $AFZR$, the pressure at any point F , arising from the incumbent weight, is in direction of the tangent TFt , the tangent at F .

For (Fig. 6.) it was shewn in Cor. 3. Prop. I. that the direction of the force at F is in direction of

Fig. of the line FB. And when the lines BF, FH
13. &c. are diminished ad infinitum, and come into the
form of a curve (Fig. 13); the direction of the
pressure is still in direction of the particle of the
curve at F, that is in the tangent TFt.

Cor. 1. *The weight of the part of the arch gives
the pressure along the tangent TFt, and the lateral or
horizontal pressure in direction IF; are respectively
TI, TF and IF.*

For the weight acts in direction TI, and the pressure
in the tangent acts in direction TF, and the
lateral pressure acts against the plain TI in direction
FI perp. to that plane. And these forces are as the
three lines TI, TF and IF.

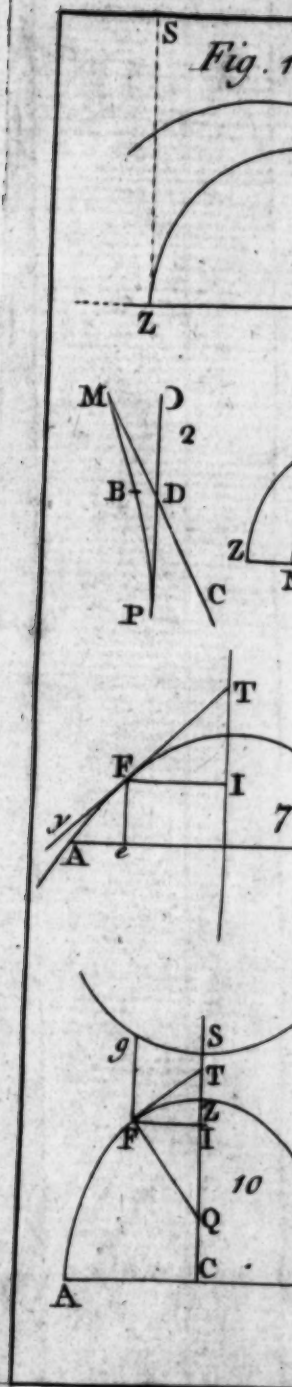
Cor. 2. *If the arch is not truly in equilibrio; and
if there is a defect of pressure at F; then the pressure
of the arch ZF at F, will be in a direction higher
than the tangent Ft. But if there is an excess of
pressure at F; the pressure at F will be lower than the
tangent Ft.*

This follows from the composition of motion.

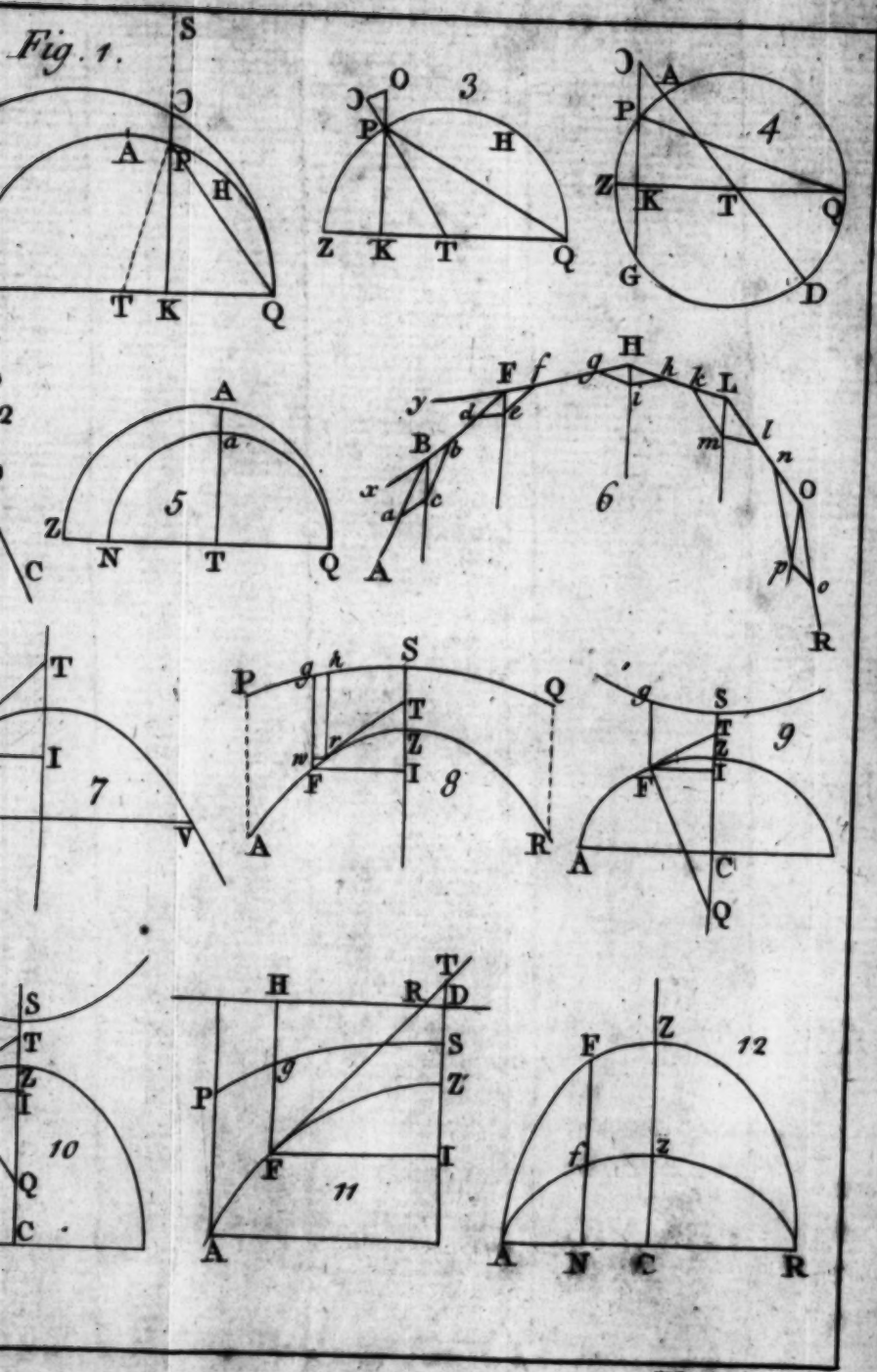
PROP. VI.

*In any arch of a bridge, &c. the joints in every
place F, must be perpendicular to the curve of the arch.*

14. Every body acting against a plain surface, acts
in lines perpendicular thereto; but if a body presses
obliquely against a plane, it will endeavour partly
to slide along that plane. And if two surfaces be
pressed together by an oblique force, these surfaces
will endeavour, according to the degree of obliquity,
to slide by one another. Therefore to prevent
that effect, these surfaces must be pressed together
by a force which acts perpendicular thereto. Hence,



Miscellanies



CF be a joint where the surfaces of two arch Fig. 14. meet, and as BA is the direction of the force; therefore CF must be perp. to the arch AB, at F; that AC and CB may remain at rest.

or. Hence if the joints be not perpendicular to the plane, there will arise a lateral pressure, whose direction is not along the tangent; and wanting a force to resist it, will destroy the equilibrium; and some of the stones will endeavour to fly out.

S C H O L.

That has been said of a linear arch, must be understood of arches of any thickness, or such as are contained between two parallel plain arches; as we see in churches, bridges, &c. These may be supposed to be generated by the motion of a linear arch, and its plane, in direction of a line perpendicular to that plane. Likewise what is said of the section at Fg, must be understood of the section of the arch passing through Fg, and perpendicular to the plane of the arch.

It must observe, that the tops of the arch-stones must be formed as you see in the figure, one part of each being parallel, and another part perpendicular to the horizon.

P R O P. VII.

Let AFZR be an arch of equilibration, and APQR 15. a wall or incumbent weight. Then if all the ordinates ZS, AS, AP, RQ, &c. be divided at D, E, B, b, &c. by a given ratio; and the arch bDEB be drawn through all the points of division D, E, &c. Then the weight ABDbR will also keep the arch AFZR in equilibrio.

For the points at Z, F are kept in equilibrio, by the weights of the wall or arch at Z, F; that is,

M

by

Fig. by ZS and Fg, and ZD, FE being in the same ratio as ZS and Fg, the same equilibrium must remain.

Or it may be considered thus; since all the parts of the wall keep all the points of the arch in equilibrium; let ZD, FE, be $\frac{1}{2}$ or $\frac{1}{3}$ &c. of ZS, Fg, and then let the wall DEFZ be twice or thrice &c. as dense matter, as SgFZ; then it is plain there will be the same pressure upon every point as before; and the arch will still be kept in equilibrium.

Cor. Hence if the figure of the arch AFZ be given, and also the figure at top SgP; we can find another curve DEB, nearly resembling any figure we please (if possible), which shall still preserve the equilibrium. And that is by dividing ZS, Fg, in such a ratio at D and E, that the points D, E, may fall in the figure required.

For an infinite number of curves DE may be drawn by such a division; and some or other of them will nearly represent the figure we want, where the thing is possible.

EXAMPLE. I.

16. Let AFZ be a circle, Sg the curve at top, found by Ex. 1. Prop. II. To find the position of the horizontal right line DE, so that the weight ZDE may keep the part of the circle ZF in equilibrium for it is impossible to do it for the whole circle.

Let AF be 30 degrees, $CF=r$, $CI=c$, $ZS=ZD=z$. Then $Fg = \frac{ar^3}{c^3}$. Then $CI=IZ$, $=\frac{1}{2}r+z=FE$, whence $ZD:ZS::FE$ or $IZ:Fg$; that is, $z:a::\frac{1}{2}r+z:\frac{ar^3}{\frac{1}{2}r^3}$ or $8a$; therefore $8az = \frac{1}{2}ar + az$, or $7z = \frac{1}{2}r$, whence $z = \frac{1}{14}r$.

E X A

EXAM. 2:

Let the arch AZR be the catenary whose latus rectum is r . To erect the wall ABbR upon it, so that all the parts of the arch may remain in equilibrio, and that the top or curve BEDF may have a proper degree of ascent. 17.

Make $ZS = r$ in the axis CZ, and through S draw the horizontal line PSQ. Then the solid APQR will keep all the parts of the arch in equilibrio. Assume a point D at pleasure, and make $ZS : ZD :: Fg : FE :: AP : AB$. Draw BED for the top. If the ascent through BED be too little or too great; it is too near PQ or AZ; and when a new point D must be taken nearer Z or S, respectively; and the points E, R, found anew; till at last the curve BED be such as was required.

PROP. VIII. PROB.

To investigate and construct the curve or arch AFZ, 18.
of equilibration; having given the points A, Z, C, and ZS the distance of the horizontal line SP; so that the weight APSZ may keep the arch AZ in equilibrio.

By Cor. 2. Prop. II. Fg is as $\frac{1}{R \times S.FTI}$ (Fig.

.) But $S.FTI = \frac{FI}{FT} = \frac{\dot{y}}{\ddot{z}}$, and $R = \frac{\dot{z}^2}{\dot{y}\ddot{x}}$ when

is given; therefore $R \times S.FTI = \frac{\dot{z}^2}{\dot{y}\ddot{x}} \times \frac{\dot{y}}{\ddot{z}} =$

and Fg is as $\frac{1}{R \times S.FTI}$ or as $\frac{\ddot{x}}{\dot{y}^2}$.

Put $ZS = a$, $ZI = x$, $FI = y$, (Fig. 18.) then Fg

at $a+x$ is as $\frac{\ddot{x}}{\dot{y}^2}$, or $a+x = \frac{rr\ddot{x}}{\dot{y}^2}$, and $rr\ddot{x} = ay^2 +$

Fig. xy^2 , assuming r constant. Therefore $rr\ddot{x}\ddot{x} = a\dot{y}^2$

$$18. + x\dot{x}\dot{y}^2, \text{ and the fluent is } \frac{rr\dot{x}^2}{2} = a\dot{x}\dot{y}^2 + \frac{x\dot{x}}{2}\dot{y}^2$$

$$\text{or } rr\dot{x}^2 = \overline{2ax + xx} \times \dot{y}^2, \text{ whence } \dot{y} = \frac{r\dot{x}}{\sqrt{2ax + xx}}$$

$$\text{and the fluent is } y = 2rL \times \log. \frac{\sqrt{x} + \sqrt{2a + x}}{\sqrt{2a}}$$

$$\text{or } y = rL \times \log. \frac{a + x + \sqrt{2ax + xx}}{a}, \text{ which is the}$$

the ordinate of the catenary, as r to a . See Ex. 12. Prop. XIII. Fluxions. Hence assuming x successively 1, 2, 3, 4, &c. the correspondent values of y will be found, which may be put into a table

Otherwise thus.

Since $rr\ddot{x} = a\dot{y}^2 + x\dot{y}^2$, assume

$$x = Ay^2 + By^4 + Cy^6 + Dy^8 \text{ \&c.}$$

$$\text{then } \dot{x} = 2Ay + 4By^3 + 6Cy^5 + 8Dy^7 \text{ \&c.}$$

$$\text{and } \ddot{x} = 2A + 3.4By^2 + 5.6Cy^4 + 7.8Dy^6 \text{ \&c.}$$

putting $\dot{y} = 1$.

$$\text{Whence } rr\ddot{x} = 2rrA + 3.4rrBy^2 + 5.6rrCy^4 + 7.8rrDy^6 \text{ \&c.}$$

$$\begin{aligned} &= a\dot{y}^2 + x\dot{y}^2 \\ &\quad \quad \quad = a + Ay^2 + By^4 + Cy^6 \text{ \&c.} \end{aligned}$$

And equating the co-efficients $A = \frac{a}{2rr}$, $B =$

$$\frac{A}{3.4rr}, C = \frac{B}{5.6rr}, D = \frac{C}{7.8rr}, \text{ \&c. whence } x =$$

$$\times \frac{yy}{rr} + \frac{a}{2} \times \frac{1}{3.4} \times \frac{y^4}{r^4} \text{ \&c. put } p = \frac{yy}{rr}, \text{ then}$$

$$x = \frac{a}{1.2} p + \frac{a}{1.2.3.4} pp + \frac{a}{1...6} p^3 + \frac{a}{1...8} p^4 \text{ \&c.}$$

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Or putting A, B, C, D, for the preceding terms, Fig.

$$= \frac{a}{1.2} p + \frac{p}{3.4} A + \frac{p}{5.6} B + \frac{p}{7.8} C + \frac{p}{9.10} D \text{ \&c. } 18.$$

D &c.

To find r , put $x = ZC = 30$ feet, $y = AC = 30$

feet, $a = 3\frac{1}{2}$ feet, then $\frac{x}{a} = \frac{p}{1.2} + \frac{p}{3.4} A + \frac{p}{5.6}$

$+ \frac{p}{7.8} C \text{ \&c. } = \frac{30}{3\frac{1}{2}} = 8.57143$, and by rever-

sion of series $p = 8.69774 = \frac{yy}{rr} = \frac{900}{rr}$, whence

$= \frac{900}{p} = 103.4754$, which being known, put

successively equal to 0, 1, 2, 3, 4, &c. from

which the several values of p will be found, by the

equation $\frac{yy}{rr} = p$; and from these the respective

values of x by the series above.

E X A M.

Let $y = 10$, then $p = \frac{100}{103.475} = .96642$, then

$$x = \frac{ap}{2} = 1.69130$$

$$+ \frac{Ap}{3.4} = .13621$$

$$+ \frac{Bp}{5.6} = .00439$$

$$+ \frac{Cp}{7.8} = .00007$$

$$1.83197 = x$$

Fig. By this method I computed the following Table,
18. and from this Table I made that which I published
in my book of *Mechanics*. Ex. 106.

values of y .	values of p .	values of x .	y .	p .	x .
feet 1	0.00967	0.0169	16	2.47402	5.2991
2	.03867	0.0678	17	2.79294	6.1366
3	.08699	0.1533	18	3.13120	7.0673
4	.15463	0.2741	19	3.48877	8.1001
5	.24161	0.4314	20	3.86567	9.2451
6	.34792	0.6267	21	4.26190	10.5136
7	.47356	0.8619	22	4.67746	11.9174
8	.61853	1.1393	23	5.11234	13.4704
9	.78280	1.4616	24	5.56655	15.1875
10	.96642	1.8319	25	6.04010	17.0855
11	1.16938	2.2537	26	6.53298	19.1825
12	1.39165	2.7312	27	7.04518	21.4788
13	1.63326	3.2690	28	7.57670	24.0569
14	1.89420	3.8722	29	8.12755	26.8815
15	2.17444	4.5467	30	8.69774	30.0000

I also computed another Table upon the same principles, making a or $ZS=7$ feet; and this Table is also published in the last edition of the *Mechanics*. By the same method other Tables may be computed for any other values of a . I shall here set down the values of p and rr for some other values of a .

If $a = 4$ feet, then $p = 8.007403$, and $rr = 112.396$;

$a = 5$, $p = 6.937512$, $rr = 129.7295$.

$a = 6$, $p = 6.139939$, $rr = 146.5813$.

$a = 7$, $p = 5.518200$, $rr = 163.0970$.

$a = 24$, $p = 2.104166$, $rr = 427.7229$.

$a = 28$, $p = 1.842124$, $rr = 488.5665$.

Cor. 1. When $r=a$, this curve becomes the catenary, whose latus rectum is a . Fig. 18.

For then they have every where the same ordi-

ates, $aL \times \log. \frac{a+x+\sqrt{2ax+xx}}{a}$.

Cor. 2. As r : to $\sqrt{2ax+xx}$:: rad. tan. angle FAC.

For rad. : tan. FAC :: $y : \dot{x} :: r : \sqrt{2ax+xx}$; here $x=ZC$, and the like for the angle at F, then $ZI=x$.

Cor. 3. If $ZI=x$, the radius of curvature at F is

$$\frac{rr+2ax+xx^{\frac{3}{2}}}{r \times a+x}$$

For this rad. $= \frac{\dot{z}^2}{y\ddot{x}}$; and $y = \frac{r\dot{x}}{\sqrt{2ax+xx}}$,

hence $\dot{x} = \frac{\sqrt{2ax+xx}}{r}$ (putting $y=1$), and $\ddot{x} =$

$$\frac{\dot{x}}{r} \times \frac{r}{a+x} = \frac{\dot{x}^2}{y} \times \frac{r}{a+x} = \frac{\dot{x}^2 + \dot{y}^2}{y} \times \frac{r}{a+x}$$

$$\frac{rr}{a+x} = \frac{2ax+xx}{rr} + 1 \times \frac{rr}{a+x} = \frac{rr+2ax+xx^{\frac{3}{2}}}{r \times a+x}$$

Cor. 4. Hence an arch of equilibration AZR may be found of any given height and breadth CZ, AR; and to have any degree of ascent BED. 17.

For instead of taking $AC=CZ$ as before, take them equal to any given numbers, and proceed as before. Thus if $x=20$, and $y=30$, $a=5$. Then

$$\frac{20}{5} \text{ or } 4 = \frac{p}{1.2} + \frac{p}{3.4} A + \frac{p}{5.6} B \text{ \&c. and by re-}$$

version p is had, then $rr = \frac{yy}{p} = \frac{900}{p}$, and then

M 4

each

Fig. each value of y is found as before. And the point D is found by Prop. VII. for the ascent BED.

Or having the curve constructed, where the height and half the breadth are equal, as in this Prop. a segment AZR may be cut off by trials, whose height to the breadth is in any given ratio, less than 1 to 2.

PROP. IX. PROB.

19. Let AFZR be an arch supporting a fluid APQR, so that all the parts F of the arch shall remain in equilibrio; gSQ being a horizontal right line. To find the nature of the curve ZF.

Let $ZS = a$, $SI = x$, $FI = y$, $ZF = z$. Then since the pressure of the fluid at F, against the arch is as the depth Fg; and the strength at F is as the curvature at F; therefore the height Fg is as the curvature at F, or reciprocally as the radius of curvature, and that is reciprocally as $\frac{\dot{z}\dot{x}}{-\dot{y}}$, $\dot{z}$ being

given; that is $\frac{-\dot{y}}{\dot{z}\dot{x}}$ is as x , or $\frac{-rr\dot{y}}{2} = \dot{z}xx$ (al-

suming the given quantity $\frac{rr}{2}$); and taking the

fluent, $-rr\dot{y} = xx\dot{z}$, and by correction $rr\dot{z} - rr\dot{y} = xx - aa \times \dot{z}$, and $rr + aa - xx \times \dot{z} = rr\dot{y}$, or (putting $rr + aa = bb$) $bb - xx \cdot \dot{z} = rr\dot{y}$, and $bb - xx \times \dot{z}^2 = \dot{b}\dot{b} - \dot{x}\dot{x}^2 \times \dot{z}^2 + \dot{y}^2 = r\dot{y}^2$, which reduced,

gives $\dot{y} = \frac{bb - xx \times \dot{z}}{\sqrt{r^4 - b^4 + 2bbxx - x^4}}$, for the nature of

the curve; and y may be found by infinite series.

SCHOL.

SCHOL.

Fig.

From hence, if a fluid could be made to stand on an arch, the curvature at F being as the height Fg, must be greater at the bottom and less at the top of the arch. But if the fluid should be congealed into ice or some solid body, the pressure would not be the same; for the pressure would then be directed downward, and not directly against the arch.

PROP. X.

If AZFR be an arch supporting the wall APQR, which wall consists of perpendicular columns Fg, and at each of these stand upon the part of the curve; and was only kept from sliding down the arch by the next adjoining column. I say the curvature at F, must be as the height Fg, that all the parts may remain in equilibrio.

Let $ZI = x$, $IF = y$, $ZF = z$. Then since $\dot{z}$ is given, the weight of the column Fog will be $\dot{y} \times Fg$. Draw Fs perp. to the curve, and sr perp. to IF or perp. to the horizon. Then og will represent the weight of og , rF its pressure against Fg, and Fs the pressure against the curve. Hence the weight ($\dot{y} \times Fg$) : pressure against the curve :: rs : Fs :: $\dot{y} : \dot{z}$. Therefore the pressure against Fo = $\frac{\dot{z} \times \dot{y} \times Fg}{y} = Fg \times \dot{z}$. But the curvature at F must be as that pressure, that is as $Fg \times \dot{z}$; that is, the curvature must be as Fg.

Cor. Hence the curve ZF is the same as that in the last Prop. For it comes to the same case as in the last Prop.

PROP.

PROP. XI.

22. *If ZFA be an arch supporting the wall ZS, which is made of arch stones GFfg lying aslope, so that all the joints GF, gf, may be perpendicular to the curve of the arch ZFf. To find the length FG of any arch stone, in any place; so that the whole arch may rest in equilibrium.*

Let Z be the vertex of the inner curve of the arch, ZS the height of the wall, HZ the rad. of curvature at Z, OF the radius of curvature at F. Take $CB = \sqrt{HS^2 - HZ^2}$, and through B draw BE parallel to the horizontal line CA. And supposing Ff, the base of the stone FGgf, to be a small part of the arch. Draw the radii FO, fO, and CD, Cd parallels thereto; then the angle DC = angle GOg. *Note*, the triangle CBD may be taken more commodiously out of the figure.

The stone FGgf is kept in equilibrium by its weight and the pressure of the two plains GF, gf. But three forces sustaining one another are as three lines drawn perpendicular to their directions; therefore if Dd represent the gravity of the stone, DC will be the pressure against the joint GF, and dC the pressure against the joint gf; and the same dC must also be the pressure against the next stone, since action and re-action are equal. And for the same reason, DC must be the pressure of the next stone towards Z, against the joint GF, to preserve the equilibrium, &c.

The sectors OGg, OFf, and CDd, as also the fluxions thereof, are as OG^2 , OF^2 , and CD^2 . Whence $OG^2 - OF^2 : CD^2 :: \text{area GFfg} : \text{triangle DCd}$. And in the vertex it is $HS^2 - HZ^2 : CB^2 :: \text{area SP} : \text{triangle CBb}$. But by construction $HS^2 - HZ^2 = CB^2$; therefore area SP = triangle CBb. And

V. OF ARCHES.

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consequently, to preserve the equilibrium, the Fig.

$FGgf = \text{triangle } CDd$, and $OG^2 - OF^2 = 22$.

and $OG^2 = OF^2 + CD^2$.

18.

Therefore to construct the exterior curve, find the center of curvature O for any given point on the curve F, and draw FO, and CD parallel to FO, and produce OF to G, and make $FG = \sqrt{OF^2 + CD^2} - OF$, and G is a point in the curve. And thus as many points may be found as you please, and the curve SGg drawn through all these points will be the top of the arch required. An arch thus constructed will remain in equilibrio, as long as the arch stones have liberty to slide down their inclined planes on which they lie.

EXAM. 1.

Let AZR be the arch of a circle; and let ZS be 21.

height at the vertex Z, and make $CB =$

$\sqrt{HS^2 - HZ^2}$, and draw BE parallel to AR. From

center C draw CF, and produce it to G, so that

CG may be equal $\sqrt{CF^2 + CD^2} - CF$. Then G

will be in the curve. If $CZ = 8$, $ZS = 1$,

$\angle ZCF = 45^\circ$; then $CD = 5.83$, $FG = 1.9$.

EXAM. 2.

Let ZFA be an ellipsis, whose latus rectum is l. 22.

Take any point F, at which point suppose a tan-

gent to the curve to be drawn, and draw FO per-

pendicular to the tangent, and whose length FO =

l. And from O, draw OFG, Ofg; and let S

be the top of the arch. Take $CB = \sqrt{HS^2 - HZ^2}$,

and draw BE parallel to CA, and CD parallel to

CF; and make $FG = \sqrt{OF^2 + CD^2} - OF$, and

G will be a point in the curve.

If $AC = 8$, $CZ = 6$, $ZS = 1$, and $ZH = 10.66$,

$CH = 11.66$, whence $CB = 4.73$; then if the ordi-

nate

CONSTRUCTION

Fig. nate at F, perpendicular to ZC be = 5, then
 22. subtangent to the axis ZC, is = 2.99; whence
 angle BCD = 30 53, $\angle$ FIA = 59 7, and FI =
 5.46, and $FO = \left(\frac{4FI^3}{l}\right) 8.06$, where $l = 9$, the
 latus rectum of AC, and $CD = 5.51$, whence at
 $FG = 1.71$.

EXAM. 3.

If ZFA be a parabola whose latus rectum is
 $CZ = 4$, $ZS = 1$, $CA = 4$. Having taken any point
 F; by the help of the abscissa and ordinate, find
 the subtangent, and the angle the tangent makes
 with the ordinate, which is the same angle the per-
 pendicular FI produced makes with the axis ZC.
 suppose the subtangent and ordinate to be equal,
 that angle will be 45 degrees, and $BCD = 45$ deg.
 let R = radius of curvature at $F = 2\sqrt{32}$, radius
 of curvature at $Z = 4$, whence $CB = 3$, $CD = \sqrt{10}$.
 And $FG = \sqrt{RR + CD^2} - R = FG = .77$.

EXAM. 4.

Let ZF be the catenary, and let the tangent at
 F make an angle of 45, with the axis, and FI will
 make the same angle, and in this case $ZF = a$, let
 $a = 6$, $ZS = 1$, radius of curvature in $Z = a = 6$,
 whence $CB = \sqrt{13}$, and angle BCD also = 45,
 therefore $CD = \sqrt{26}$. And radius of curvature in

$F = a + \frac{zz}{a} = 2a = 12$, because $z = a$. Whence

$$FG \sqrt{144 + 26} - 12 = 1.04.$$

EXAM. 5.

Let ZF be a cycloid, the diameter of the gener-
 ating circle $CZ = 6$, $ZS = 1$, rad. curvature in
 $Z = 2CZ = 12$, and suppose the ordinate drawn to
 F, to pass through the center of the generating cir-
 cle.

then the tangent at F, and also the perp. GI Fig. 22. cut the axis ZC at an angle of 45; in which 22. the radius of curvature at F = $\sqrt{2CZ^2} = 8.48$, and CB = 5, CD = $\sqrt{50}$. Whence $DE = \sqrt{122} - 8.48 = 2.56$. This is the best we tried yet for the figure of an arch of this sort.

SCHOL.

None of the three cases mentioned in the three propositions can hold good in the construction of arches. For the wall is not in the nature of a solid (Fig. 19.) nor do the columns endeavour to fall down in direction of F (Fig. 20.) or in direction of G (Fig. 21.) For when the parts are cemented together by the mortar, they will all act perpendicularly to the horizon, and are each fixed to the parts of the arch, as supposed in Prop. III.

What has been demonstrated in Prop. VII. in regard to the arch AZR (Fig. 15.) holds true in these arches also, only taking the joints instead of perpendicular lines. That is, if ZS, FG, fg, and the other joints, be divided in any given ratio, and a curve line drawn through all the points, for the top of the arch; the arch will still remain in equilibrium. Or if the joints be increased in any given ratio, the line drawn through all the points will be the top of the arch, which will still be in equilibrium. See also Prob. 58. Art. 18.

PROP. XII.

If SM be a line parallel to the horizon, and ZQ a curve of such a nature, that drawing LQM, and LN parallel to it; so that LQ be the radius of curvature at Q, and $2LQ + QM \times QM$ be always equal to CN^2 ; then all the parts of the arch ZSMQ will remain in equilibrium.

For

Fig. For by Prop. XI. if $CB = \sqrt{HS^2 - HZ^2}$,
 22. $LM^2 - LQ^2 = CN^2$, then the arch ZQ will be
 in equilibrio. But $LM = LQ + QM$, and $LM^2 - LQ^2 = LQ^2 + 2LQ \times QM + QM^2 - LQ^2 = 2LQ \times QM + QM^2 = CN^2$, that is, $2LQ \times QM + QM^2 = CN^2$,
 therefore all the parts of the arch bounded by the
 right line SM and the curve ZQ will remain
 equilibrio.

But the calculation of such a curve is so very
 intricate, that it can be of no manner of use in the
 building of bridges.

PROP. XIII.

23. If any curve AFZ be made to revolve round its
 ZC, and form thereby a concave surface or vault;
 if a wall SZFg be erected thereon, which is to be
 supported by that surface, like an arch. Then the
 weight of that wall at any point F, so that all the parts
 stand in equilibrio, that is Fg, will be as $\frac{FT^2}{FI \times FI}$
 where FT is a tangent at F, FI an ordinate, and
 the radius of curvature in F.

Let ZAC, ZaC be two planes, comprehending
 the small part of the vault ZAa; then the distance
 Ff at any place F will be as the ordinate FI; and
 the incumbent weight on the line Ff will be $Fg \times Ff$
 or as $Fg \times FI$.

Let $ZI = x$, $FI = y$, $ZF = z$, and let z be a given
 part of the curve ZF; and y a small part of the
 ordinate FI. Then $y = \frac{z \times FI}{FT}$, and y is as $\frac{FI}{FT}$
 because z is given. Then the incumbent weight
 on the small part of the surface at F is as Fg

$I \times \dot{y}$ or as $Fg \times \frac{FI^2}{FT}$. By Prop. II. the pressure

at any point F is as $\frac{C = \text{curv.}}{S. ITF^2} = C \times \frac{FT^2}{FI^2}$. Whence

$Fg \times \frac{FI^2}{FT}$ is as $C \times \frac{FT^2}{FI^2}$ or as $\frac{FT^2}{FI^2 \times R}$, and Fg as

$$\frac{FT^2}{FI^2 \times R}$$

Cor. 1. Hence Fg is as $\frac{-\dot{x}\ddot{y}}{y\dot{y}^3}$, putting $\dot{x}$ invariable.

For $\frac{FT}{FI} = \frac{\dot{z}}{\dot{y}}$, and $R = \frac{\dot{z}^2}{-\dot{x}\ddot{y}}$; therefore Fg

being as $\frac{FT^2}{FI^2 \times R}$ is as $\frac{\dot{z}^2}{\dot{y}^2 y R}$ or as $\frac{\dot{z}^2}{y\dot{y}^3} \times \frac{-\dot{x}\ddot{y}}{\dot{z}^2} =$

$\frac{-\dot{x}\ddot{y}}{y\dot{y}^3}$, as is also shewn in my book of Fluxions.

or thus, $R = \frac{\dot{z}^2}{y\ddot{x}}$ when $\dot{y}$ is given, whence Fg is as

$$\frac{\dot{y}\ddot{x}}{\dot{z}^2} = \frac{\ddot{x}}{y\dot{y}^2}.$$

Cor. 2. Hence no curve can produce the figure of a true vault, except the radius of curvature in the vertex be infinite.

For in the vertex, $FT = FI$, and Fg in the vertex

is $\frac{FI^2}{FI^2 \times R} = \frac{1}{FI \times R} = \text{infinity}$, when R is finite, and $FI = 0$, therefore to make Fg a real quantity, R must be infinite; otherwise ZS will be infinite.

EXAM. I.

Let FZR be the common parabola, $rx = yy$, then 24.

$FT = 2x$, and $FT = \sqrt{yy + 4xx} = \sqrt{rx + 4xx}$, and

$$R =$$

Fig. $R = \frac{rr + 4yy^{\frac{1}{2}}}{2rr}$. Therefore Fg is as $\frac{rx + 4xx^{\frac{1}{2}}}{y^4 R}$
 as $\frac{rx + 4xx^{\frac{1}{2}}}{rrxx} \times \frac{2rr}{rr + 4rx^{\frac{1}{2}}}$ that is, Fg is as $\frac{r}{r^{\frac{1}{2}}}$
 or as $\frac{1}{\sqrt{x}}$, or Fg is as $\frac{1}{y}$; and therefore at the
 vertex Fg is infinite. And so it will in the circle.

E X A M. 2.

25. Let AFZR be a cubic parabola, whose equation
 is $x = y^3$. Here $IT = 3x$, and $FT = \sqrt{yy + 9y^6}$
 $= \sqrt{yy + 9y^6}$. And $R = \frac{1 + 9y^{\frac{1}{2}}}{6y}$, whence Fg

as $\frac{FT}{FI + R}$ or as $\frac{yy + 9y^6}{y^4} \times \frac{6y}{1 + 9y^{\frac{1}{2}}} = \frac{6yx^{\frac{1}{3}}}{y^{\frac{1}{2}}}$

as 6 a given quantity. Therefore Fg is every where
 the same, and the curve Sg is also a cubic parabola

the same with the former, but in a higher position.
 After the same manner if AFZ be a biquadratic
 parabola, where $x = y^4$, there will come out Fg as
 y ; and therefore the height at the vertex is 0.

P R O P. XIV. PROB.

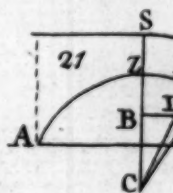
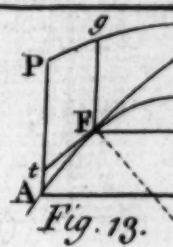
26. To investigate and construct the curve or arch AFZ,
 by whose revolution a vault AZR is formed, upon
 which the wall or building APQR being erected, and
 being bounded by the horizontal line PQ, the height
 ZS being given; all the parts may remain in
 equilibrio.

Draw ZG parallel to QSP, and let $SZ = a$,
 $ZI = x$, $FI = y$, $AC = b$, $ZC = b$. Then $Fg = a + b$

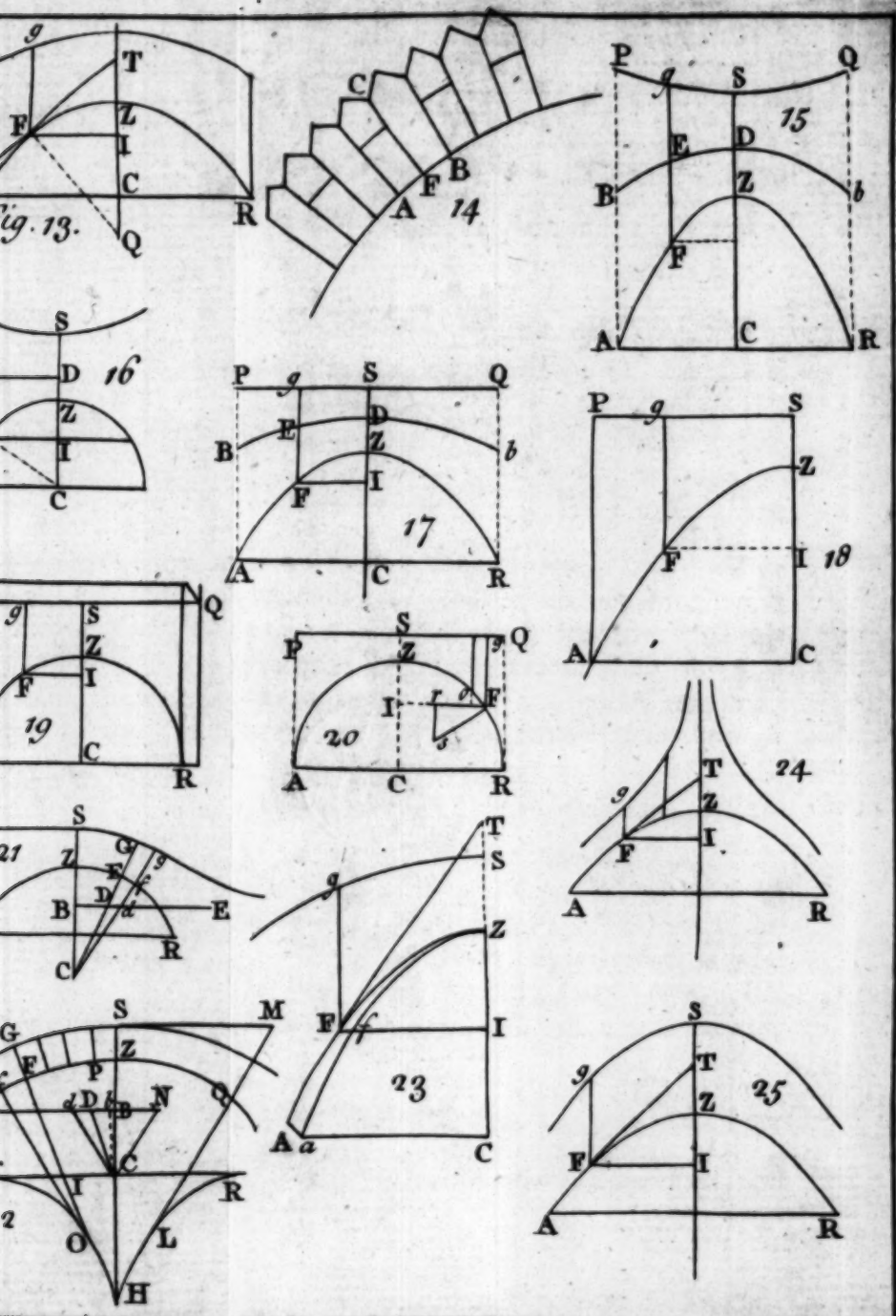
but (by Cor. 1. Prop. XIII.) Fg is as $\frac{x}{yy^2}$; whence



Miscellanies



Miscellanies



as $a+x$; and taking the constant quantity r^3 , 26.

$$= cy^3 + xy^3, y \text{ being } = 1.$$

inding the indices (by Rule III. Prop. X. Sect. uxions) to be 3, 6, 9, 12, &c.

$$\begin{aligned} x &= Ay^3 + By^6 + Cy^9 + Dy^{12} \&c. \\ \dot{x} &= 3Ay^2 + 6By^5 + 9Cy^8 + 12Dy^{11} \&c. \\ \ddot{x} &= 2.3Ay + 5.6By^4 + 8.9Cy^7 + 11.12Dy^{10} \&c. \\ r\ddot{x} &= 2.3r^1Ay + 5.6r^3By^4 + 8.9r^3Cy^7 + 11.12r^3Dy^{10} \&c. \\ &= ay + Ay^4 + By^7 + Cy^{10} \&c. \end{aligned}$$

$$\text{equating the co-efficients } A = \frac{a}{2.3r^3}, B =$$

$$C = \frac{B}{8.9r^3}, D = \frac{C}{11.12r^3}, \&c. \text{ put } p = \frac{y^3}{r^3}$$

$$\text{hence } x = \frac{ay^3}{2.3r^3} + \frac{ay^6}{2.3.5.6r^6} + \frac{ay^9}{2.3.5.6.8.9r^9}$$

$$= \frac{a}{2.3} p^1 + \frac{a}{2.3.5.6} p^6 + \frac{a}{2.3.5.6.8.9} p^9 \&c.$$

putting A, B, C, D, &c. for the preceding

$$= \frac{a}{2.3} p + \frac{p}{5.6} A + \frac{p}{8.9} B + \frac{p}{11.12} C + \frac{p}{14.15} D \&c.$$

et SZ or $a=1$, $x=b=10$, $y=b=15$. Then

$$\text{nd } r, \text{ we have } \frac{b}{a} = \frac{p}{6} + \frac{p}{30} A + \frac{p}{72} B +$$

C &c. = 10. And by reverſion of ſeries $p =$

$$= \frac{b^3}{r^3}, \text{ and } r^3 = \frac{b^3}{26.37} = \frac{3375}{26.37} = 128.$$

nd r^3 being known; put $y=0, 1, 2, 3, 4, \&c.$

which find the ſeveral values of p by the equa-

$$\frac{y^3}{r^3} = p, \text{ and from theſe the correſpondent va-}$$

of x by the foregoing ſeries.

Fig.
26.*Easier thus.*

Instead of assuming some value for a , assume the value of r^3 , then the equation $\frac{b^3}{r^3} = p$, gives p which being known, the series $\frac{p}{6} + \frac{p}{30} A + \frac{p}{72} B$ &c. gives $\frac{b}{a}$; whence a will be known. But if a is not of a commodious length, assume r^3 anew till a be of a proper magnitude.

For example, take $r^3 = 130$, then $p = \frac{3375}{130} = 25.9615$, in this case. Then $\frac{b}{a} = \frac{1}{6} p + \frac{1}{30} p A + \frac{1}{72} p B + \frac{1}{132} p C + \frac{1}{210} p D$ &c. $= 9.72285$, and $a = \frac{10}{9.72285} = 1.02851$. Now put $p = \frac{y^3}{r^3}$, and assume $y = 0, 1, 2, 3, 4$, &c. successively, then so many values of p will be had, and then the corresponding values of x , by the series $x = \frac{a}{2.3} p + \frac{p}{5.6} A + \frac{p}{8.9} B + \frac{p}{11.12} C$ &c. And these being all computed will be as in the following Table.

values of y .	values of p .	values of x .	values of y .	values of p .	values of x .
1	.00769	.0013	9	5.60770	1.1555
2	.06154	.0106	10	7.69231	1.6950
3	.20769	.0358	11	10.23846	2.4461
4	.49231	.0858	12	13.29231	3.4945
5	.96154	.1702	13	16.90000	4.9652
6	1.66154	.3009	14	21.10769	7.0425
7	2.63846	.4935	15	25.96154	10.0000
8	3.93846	.7687			

It is easy to construct this curve by the Table, setting off ZD or y by the first column; and DF or p by the third; and drawing a curve through all the points. The curve is very flat at the top, and at the bottom rises at an angle of about $71\frac{1}{2}$ degrees.

Fig.

A R T. VI.

The Precession of the Equinoxes.

IN Prob. XXVI. of my book of Fluxions, I have given a calculation of the Precession of the Equinoxes, *ab origine*, that is, depending upon nothing but the laws of motion. What I give here is deduced primarily from the motion of the moon's nodes, which makes the calculations vastly shorter.

P R O P. I. PROB.

27. *Supposing the density of the earth the same throughout, to find the precession of the equinoxes, by the sun's force.*

Let QAR be the earth, CA its radius, and put
 a = earth's radius CA,
 $b = 72706''$, motion of the moon's nodes in a year,
 $n = 1436'$, a syderial day,
 $d = 40217'$ the moon's periodic time,
 $e = 52441$ the matter in the earth,
 $l = 459$ the matter in the crust,
 $x = \cos. 5^\circ$, and $c = \cos. 23^\circ \frac{1}{2}$, f = generating force.

By Cor. 3. Prop. 28. *Centripetal Forces*, $d : n :: b :$
 $\frac{bn}{d}$ the yearly motion of the nodes of a moon, revolving in the time n , or of an equinoctial ring.

Now by Prop. 59. *Mechanics*, the same angular motions will be produced, if the bodies of the ring, and of the earth, be placed in their centers of gyration,

ART. VI. OF EQUINOXES.

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variation, or at the distances $a\sqrt{\frac{1}{2}}$ and $a\sqrt{\frac{2}{3}}$; there Fig.

are (by Cor. 2. Prop. 56. ib.) it will be as $\frac{f}{l \times \frac{1}{2}aa} : 27.$

$\frac{f}{\frac{1}{2}aa + e \times \frac{2}{3}aa} : :$ so angular motion of the equi-

noctial ring $\frac{bn}{d} : \frac{\frac{1}{2}l \times bn}{d \times \frac{1}{2}l + \frac{2}{3}e} =$ motion of the nodes

of the whole earth; that is, when the force (f) acts together upon the matter l in the equinoctial ring. But the force upon the crust or exterior part, diffused over the whole surface of the sphere, is but

$\frac{\frac{2}{3} \times \frac{1}{2}lbn}{d \times \frac{1}{2}l + \frac{2}{3}e}$ of the equinoctial force. Therefore

$\frac{lbn}{d \times \frac{1}{2}l + 2e} =$ motion of the earth's nodes, sup-

posing the equinoctial elevated above the ecliptic, more than the moon's orbit. But the force (and consequently the motion) will be less in proportion to the cosines of elevation; therefore $x : c$

motion of $e : \frac{clbn}{xdw} =$ the true motion of the equinoxes, by the sun's force; putting $\frac{5}{2}l + 2e = w$ 106030.

Operation.

$$\log. c = 9.962398$$

$$l = 2.661813$$

$$b = 4.861570$$

$$n = 3.157154$$

$$20.642935$$

$$x = 9.998344$$

$$d = 4.604409$$

$$w = 5.025429$$

$$19.628182$$

$$\log. 1.014753$$

Motion of the equinoxes $10''.345$, by the sun.

N 3

P R O P.

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Fig.
27.

PRECESSION OF

PROP. II.

Suppose the density of the earth to be reciprocal as the n^{th} power of the distance from the center; then the angular motion of the nodes in the uniform sphere is to the angular motion of the nodes in this variable sphere :: as 5 : to $5 - n$; supposing both of the same density at the surface.

Let $CA = r$, CB or $CO = x$, $c = 3.1416$, d density at A at the surface. Then $\frac{r^n d}{x^n}$ will be the density at B; and the spherical surface at B = $4\pi x^2$ and by the variable density, the matter in the orb at B = $4\pi x^2 \dot{x} \times \frac{r^n}{x^n} d = 4\pi c d r^n x^{2-n} \dot{x}$. But (by Ex

5. Prob. 23, *Fluxions*), the distance of the center of gyration of the orb BO, is $= x \sqrt{\frac{2}{3}} = Cp$, and of the sphere $r \sqrt{\frac{2}{3}}$. Therefore (by Cor. 2. Prop. 56. *Mechanics*) the force remaining the same, the angular motion of the nodes in the uniform sphere

to that in the invariable sphere :: $\frac{I}{\frac{2}{3} r r \times \text{sphere}}$

$\frac{I}{\text{sum of all } c p^2 \times 4\pi c d r^n x^{2-n} \dot{x}}$. But the sphere = $\frac{4}{3} \pi r^3$

Also the sum of all the $C p^2 \times 4\pi c d r^n x^{2-n} \dot{x}$, or of all the $\frac{2}{3} x x \times 4\pi c d r^n x^{2-n} \dot{x}$, or of all $\frac{8}{3} c d r^n x^{4-n} \dot{x} = \frac{8}{3} c d r^n \times \text{fluxion}$

of $x^{4-n} \dot{x} = \frac{8}{3} c d r^n \times \frac{x^{5-n}}{5-n}$. But when $n =$

$\frac{8}{3} c d r^n \times \text{fl. } \frac{\dot{x}}{x} = \frac{8}{3} c d r^n \times \log. x$. Whence the a

gular motion in the uniform sphere : to the a

gular motion in the variable sphere :: $\frac{I}{\frac{8}{3} c d r^n}$

$\frac{1}{x^{5-n}}$, or when $x = r$, as $\frac{1}{3}cdr^2 \times \frac{r^{5-n}}{5-n}$: Fig. 27.

$cd r^5$, or as $\frac{15r^5}{5-n} : 3r^5$, or as 5 to $5-n$.

Cor. 1. When $n = 1$, or the density reciprocally as the distance from the center; the angular motions in the uniform and variable spheres, will be as 5 to 4, for the motion of the equinoxes.

Cor. 2. If $n = 5$, the motion of the equinoxes will be nothing.

For the motion of the uniform sphere is as log. x ; but when $x = 0$, log. $x =$ infinity, and the motion as log. $x - \log. 0 = \log. \frac{x}{0} =$ infinity. And therefore the motion of the variable sphere, is 0 in respect to it.

Cor. 3. If n is greater than 5, the motion is negative or backward.

Cor. 4. If n is negative, or which is the same thing, if the density be as x^n ; then the motion of the equinoxes in the uniform and variable spheres, will be as 5 to $5 + n$.

PROP. III.

Suppose the density at the center C greater than that at the surface A, in any ratio as n to 1. The motion of the equinoxes in the uniform sphere, is to that in the variable sphere; as $n + 5$ to 6; supposing them both of the same density at the surface; and in the latter to increase uniformly from A to C.

Let $AD = d$ the density at the surface A, $CG =$ the density at the center, $BC = x$; draw BH parallel

Fig. parallel to CG or AD, then BH will be the de
28. sity at B. Draw DF parallel to AC, and FG =

$\overline{n-1.d}$. By similar triangles DF : FG :: DE : EH

that is $r : \overline{n-1.d} :: r-x : EH = \frac{r-x}{r} \times \overline{n-1.d}$

and BH = $\frac{r-x}{r} \times \overline{n-1.d} + d =$ density

B=D, by substitution. Then $4cx^2\dot{x} \times D =$ ma
ter in the orb at B. Then reasoning as before
motion in the uniform sphere, to the motion in the

variable one; as $\frac{1}{\frac{8}{15}cdr^5}$ to $\frac{1}{\text{sum of all } \frac{8}{3}xx \times 4cx^2\dot{x} \times D}$

or as sum of all the $\frac{8}{3}cx^4\dot{x} \times D$ to $\frac{8}{15}cdr^5$; but

the sum of all $\frac{8}{3}cx^4\dot{x} \times D =$ sum of $\frac{8}{3}cx^4\dot{x}D =$

fluent of $\frac{8}{3}cx^4\dot{x} \times$ by $nd - \frac{dx}{r} \times \overline{n-1} : =$

$\frac{8}{3}cndx^4\dot{x} - \frac{8cd}{3r}x^5\dot{x} \times \overline{n-1} = \frac{8}{15}cndx^5 - \frac{8cd}{18r}\overline{n-1}$

$\times x^6$. Therefore the motions in the uniform and

variable sphere, are as $\frac{8}{15}cndx^5 - \frac{8cd \times \overline{n-1}}{18r}$

to $\frac{8}{15}cdr^5$. And when $x=r$, these motions are

as : $\frac{8}{15}cnd - \frac{8}{18}cd \cdot \overline{n-1} : \times r^5$ to $\frac{8}{15}cdr^5$, or as $\frac{1}{15}$

$\overline{n - \frac{8}{15}n + \frac{8}{15}} : \frac{8}{15}$, or as : $\frac{1}{15} - \frac{1}{18} \times n + \frac{1}{15}$

$\frac{1}{15}$, that is, as $n+5$ to 6.

Cor. 1. If $n=0$, or the density be 0 at the cen
ter; the motion of the equinoctial points, in the uni
form and variable spheres; will be as 5 to 6.

Cor. 2. If $n=1$, these motions will be equal.
For then they are both uniform.

PROP. IV. PROB.

Fig.
29.

Let AR be a sphere so formed, that the decrease of density, in going from the center to the circumference, at B, may be as CB^m , that is, as any power m of the distance from the center, to find the motion of the equinoctial points.

Let $AD = d$, be the density at A, $CG = b = nd$, be the density at C; and let DHG be the curve of density. Draw BEHO parallel to CG; and let $CB = GO = x$, $FG = a$, CA or $GK = r$. Then by nature of the curve GH, GK^m or $CA^m : FG$ or $AD :: CB^m$ or $GO^m : OH$; that is, $r^m : a :: x^m$ $OH = \frac{x^m}{r^m} a$, the decrease of density at B; there-

fore $b - \frac{x^m}{r^m} a =$ density at B; and $4cx^2 \dot{x} \times b - \frac{x^m}{r^m} a$ = matter in the orb at B. And the sum of all the $\frac{2}{3}xx \times 4cx^2 \dot{x} \times b - \frac{x^m}{r^m} a$, or the fluent of $\frac{2}{3}cx^4 \dot{x}$

$\times b - \frac{x^m}{r^m} a = \frac{8}{35} cbx^5 - \frac{8cax^{m+5}}{3r^m \times m + 5}$. Whence the motions in the uniform and variable spheres are as $\frac{8}{35} cbx^5 - \frac{8cax^{m+5}}{3r^m \times m + 5}$ and $\frac{2}{15} cdr^5$; and (when $r = r$) as $\frac{1}{5} br^5 - \frac{arm+5}{r^m \times m + 5}$ to $\frac{1}{5} dr^5$, or as $b - \frac{5a}{m+5}$ to d , that is as $nd - \frac{5}{m+5} \times n - 1. d$ to d , whence the motions in the uniform and variable spheres are as $n - \frac{5 \times n - 1}{m+5}$ to 1, or as $mn + 5$ to $m + 5$.

Cor.

Fig. Cor. If $m = 1$, the motions are as $n + 5$ to 29. agreeable to Prop. III.

PROP. V. PROB.

27. Suppose the density of the earth at any place B, be as $\frac{1}{CB^n}$; going upwards from B to A; but that is hollow below the surface BFD; to find the motion of the equinoctial points in this hollow sphere; having that of a similar hollow sphere of uniform density, having the same density at A.

Let $CA = r$, $d =$ density at A, $CB = x$, proceeding as in Prop. II. The surface at B $= 4\pi x^2$ and the matter in the orb $= 4\pi x^2 \dot{x} \times d$ for the uniform sphere, and for the variable sphere $= 4\pi x^2 \dot{x}$

$\frac{r^n}{x^n} d$ or $4\pi d r^n x^{2-n} \dot{x}$; these multiplied by $\frac{2}{3} x x$ (the

square of the distance of the center of gyration) will be $\frac{8}{3} \pi c d x^4 \dot{x}$ for the uniform sphere, and $\frac{8}{3} \pi c d r^n x^{4-n} \dot{x}$ for the other. Therefore the motion in the uniform sphere : motion in the variable

:: as $\frac{I}{\text{sum} : \frac{8}{3} \pi c d x^4 \dot{x}} : \frac{I}{\text{sum} : \frac{8}{3} \pi c d r^n x^{4-n} \dot{x}} :: \text{fluent of } \frac{8}{3} \pi c d \times r^n x^{4-n} \dot{x} : \text{fluent of } \frac{8}{3} \pi c d x^4 \dot{x} :: \text{fluent of } r^n x^{4-n} \dot{x} : \text{fluent of } x^4 \dot{x} :: \frac{r^n x^{5-n}}{5-n} : \frac{x^5}{5}$, for the hollow part

BOFD. Put $x = r$, and the fluents are $\frac{r^5}{5-n}$ and

$\frac{r^5}{5}$, for the whole spheres. Subtract the respective parts from the wholes, and the motion of the

equinoxes, for the uniform and variable hollow

spheres, will be as $\frac{r^5 - r^n x^{5-n}}{5-n}$ to $\frac{r^5 - x^5}{5}$.

ART. VI. THE EQUINOXES.

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Fig.
27.

Cor. The motion in the whole uniform sphere, is to

the motion in the variable hollow sphere; as $\frac{5}{r^5}$ to

$$\frac{5-n}{r^5 \times 5-n}$$

For the motions of the uniform and variable

spheres are as $\frac{1}{fl. x^4 x}$ and $\frac{1}{fl. r^{5-n} x}$, or as $\frac{1}{\frac{1}{2} x^5}$ and

$\frac{1}{\frac{1}{2} x^5}$, or as $\frac{5}{x^5}$ and $\frac{5-n}{r^5 x^{5-n}}$; and the mot.

the whole uniform sphere, to the motion in the variable hollow sphere; as $\frac{5}{r^5}$ to $\frac{5-n}{r^5 - r^5 x^{5-n}}$

What is done here with regard to the second proposition; may as easily be done upon the suppositions laid down in the 3d and 4th propositions; and therefore I shall not trouble the reader with them. I shall only give one example of this Prop. and Corol.

Let $n=1$, $x=\frac{1}{2}r=CB$. Then the mot. in the uniform and variable hollow spheres ABFDQ, are

$$\frac{r^5 - r \times \frac{r^4}{16}}{4} \text{ to } \frac{r^5 - \frac{1}{32} r^5}{5} \text{ or as } \frac{15}{4 \times 16} \text{ to } \frac{31}{5 \times 32}$$

that is, as 75 to 62.

And the motion in the whole uniform sphere, to that in the variable hollow sphere, is as $\frac{5}{r^5}$ to

$$\frac{4}{r^5 - r \times \frac{r^4}{16}}, \text{ or as } 5 \text{ to } \frac{4 \times 16}{15} \text{ or as } 75 \text{ to } 64.$$

Hence if the motion of the equinoxes be known for a earth of uniform density, the motion of a earth formed according to any law of density will also be known, by the foregoing propositions. And

Fig. And thus I have completed that difficult problem of finding the precession of the equinoxes by the sun's force, upon any supposition of density.

S C H O L.

The moon's disturbing force is not precisely known, and we have not sufficient *data* to determine it. Let P = earth's periodical time, or year, and p = earth's periodical time about the moon at rest; then the moon's disturbing force = $\frac{PP}{p^3} \times$ sun's disturbing force. But as p is unknown we must judge as well as we can from the foregoing propositions.

Precession of the equinoxes by observation	50 ⁰ .3
in a earth of uniform density, by the sun	10.34
by Prop. 2. Cor. 1. $n = 1$.	8.27
by Prop. 3. $n = 2$.	8.86

Hence this may be concluded in general, whatever be the law of density, if it be greater within the earth, the motion by the sun's force will be less; and consequently the moon's force so much greater. Neither is there any way to make the moon's force less, but by making the earth hollow, or else rarer within, which is not probable. Hence we cannot well suppose, that the precession by the sun is greater than 10.34; nor consequently that by the moon to be less than $(50.3 - 10.34) 39.96$; then $\frac{39.96}{10.34} = 3.86$ for the moon's force, supposing the sun's force 1. But by the foregoing Props. the precession by the sun must probably be less, and therefore its force less, and therefore the moon's bigger than is here set down.

A R T. VII.

The Construction of Logarithms.

Design not here to treat of the best and most expeditious methods of making logarithms, I have already done that in my book of Fluxions. I shall here only shew how this may be done by the Indices or roots of numbers; by such methods the tables of logarithms were with great labour constructed at first. And though these methods are very tedious, and require a great deal of time and pains to perform, yet they shew the nature of logarithms, and their relation to the numbers themselves in the clearest and plainest manner.

P R O B I.

To make logarithms by the indices of the roots of 10.

This method depends on assuming some number for the logarithm of 10, and the several multiples thereof, for the logarithms of the corresponding powers of 10; and then finding the logarithms or Indices for the intermediate powers. According to this method, suppose $\log. 1 = 0$, $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, &c. where 0, 1, 2, 3, &c. are the assumed logarithms of 1, 10, 100, 1000; then to find the logarithms of the intermediate numbers, suppose an immense series of geometrical proportionals from 1 to 10, &c. Then all numbers may be supposed to be contained in this series, and every number may be some power (or root) of 10; and the index expressing this power or root, will be the logarithm of that number; and the business

Fig. 1. The fineness is to find that index or logarithm. This series may be denoted thus, 1, x , x^2 , x^3 , x^4 , x^5 , &c. to $x^{10000000} = 10$. And therefore to find the logarithm of any number is to find its place in the series. And this may be done, by help of the following Table of the square roots of 10. This Table is made by continually extracting the square roots of 10, till the indices of x all vanish, as follows.

Square roots of 10		powers of x .
	=	$x^{10000000}$
3.1622776	=	$x^{5000000}$
1.7782794	=	$x^{2500000}$
1.3335214	=	$x^{1250000}$
1.1547819	=	x^{625000}
1.0746078	=	x^{312500}
1.0366329	=	x^{156250}
1.0181517	=	x^{78125}
1.0090350	=	x^{39062}
1.0045073	=	x^{19531}
1.0022511	=	x^{9765}
1.0011249	=	x^{4882}
1.0005623	=	x^{2441}
1.0002811	=	x^{1221}
1.0001405	=	x^{610}
1.0000703	=	x^{305}
1.0000351	=	x^{152}
1.0000175	=	x^{76}
1.0000087	=	x^{38}
1.0000044	=	x^{19}
1.0000022	=	x^9
1.0000011	=	x^5
1.0000005	=	x^2
1.0000003	=	x^1
1.0000001	=	x^1

In this Table it may be observed, that any index of x , divided by 10000000, is the logarithm

its correspondent number. Thus 1 is the log. Fig. 10, and .5 is the log. of 3.1622776; and .25 the log. 1.7782794; and so on through the Table. And since the addition of these indices or logarithms, answers to the multiplication of the correspondent numbers; therefore to find the log. of any number by this Table, we must seek out such numbers in the Table, that being multiplied together (or perhaps divided), they will produce the given number; so the indices being added (or perhaps subtracted) will give the log. of that number. Therefore to find the logarithm of any number, Proceed thus. Take the nearest number thereto in the Table (on the left hand), which set down with the power or index of x against it. If it be too little, multiply by some following number in the Table, found by trials, such that the product may come the nearest to the given number; but if it was too great, divide by such a number. Set down that number and the index of x against it; then below, set down the product (or quotient) of the numbers, and against it, the sum (or difference) of the indices.

Go on the same way, by continually multiplying (or dividing) by less and less numbers, and adding (or subtracting) their indices, as you proceed, and thus you will continually approximate to the number and to its logarithm. So that at last when you have obtained the number, and its index; divide the index by 10000000, and you have the logarithm.

Note 1. To find the next multiplier expeditiously (without too many trials); divide the given number, by the last number, so far as to get a figure or two in decimals; then the quotient will point out the multiplier in the Table. Or divide, to get a divisor.

Note

Fig. Note 2. You may always use multiplication if you please, provided you never make a product exceed the number given; but the work will go on slower.

E X A M.

To find the logarithm of 2.

Num.	Indexes.
1.7782794	2500000
× 1.1547819	625000
pr. 2.0535247	3125000
÷ 1.0366329	156250
qu. 1.9809567	2968750
× 1.0090350	39062
pr. 1.9988546	3007812
× 1.0005623	2441
pr. 1.9999785	3010253
× 1.0000087	38
pr. 1.9999959	3010291
× 1.0000022	9
pr. 2.0000002	3010300

then $\frac{3010300}{10000000} = .3010300$ the log. 2.

Here the nearest number to 2 in the Table, is 1.7782794, which I set down, and the index of 2500000. Then the nearest multiplier to produce 2, is 1.1547819, which I set down and its index of x, 625000. And below I set the product 2.0535247; and the sum of the indices 3125000. Then this last number being greater than 2, I find the divisor 1.0366329, which I set down, and the index of x, 156250. And below I set the quotient 1.9809567; and the difference of the indices 2968750. The next multiplier I find to be 1.0090350, which I set down, and its index 39062. Then

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When I set down the product 1.9988546; and the index of the indices 3007812. And thus I proceed I get 2 and the index 3010300; and thence the product 3010300.

To get the multipliers as I proceed, I divide 2

1.77, and $\frac{2}{1.77} = 1.13$, and the nearest number in the Table is 1.1547819. To get the divisor

$\frac{2.053}{2} = 1.026$, and the nearest number is 1.009329. To find the next multiplier, $\frac{2}{1.9809}$

1.009, and the divisor is 1.0090350. And so to the last.

Where note, a greater number (than 2.) requires a divisor, and a less, a multiplier.

EXAM. 2.

To find the log. of a great number as 37.

First find the log. of 3.7 as before, which is .682017; and then the log. 37 is 1.5682017.

SCHOL. I.

On the contrary, the number may be found by the same Table, answering to the given logarithm; it is done by continually approximating to the given index, as before you did to the number. which is done, by continually adding (or subtracting) such indices, as to make the sum (or difference) the nearest the given index; and at the same time multiplying (or dividing) the numbers.

O

EXAM.

E X A M.

What is the number of the log. 2.5406725.
First find it for .5406725, thus

Number.	Index.
3.1622776	5000000
1.0746078	312500
3.3982080	5312500
1.0181517	78125
3.4598912	5390625
1.0045073	19531
3.4754858	5410156
1.0005623	2441
3.4735327	5407715
1.0002811	— 1221
3.4725561	5406494
1.0000351	152
3.4726785	5406646
1.0000175	76
3.4727392	5406722
1.0000005	2
3.4727410	5406724

So that 3.4727410 is the number of the log. 2.5406724; and therefore 347.27410 is the number of the log. 2.5406725. But in such cases the general Table ought to be made to more places of figures.

In this example, I take the nearest index 5000000, and its number 3.1622776. Then I take the index 312500, as the nearest to 406725, and I set it down and its number 1.0746078. Then adding the indices, we have 5312500; and multiplying the numbers, gives 3.3982080. Then I take the index 78125 (the nearest to .009422)

that is wanting to make up 5406725), and its number 1.0181517. Then I get the index 5390625 and its number 3.4598912. And so proceed to end.

SCHOL. 2.

Instead of making a Table, by taking the square roots of 10 continually, one may use other roots, the cube root, &c. But in working by it, the numbers will not converge so fast. And observe, the nearer 1, the index of the root is, the faster the numbers will converge, but there will be more labour in making the Table. Thus instead of $\frac{1}{2}$ (the index of the square root) take $\frac{2}{3}$ or rather $\frac{3}{4}$ for the index, for making the Table. In working with $\frac{3}{4}$, each number must be cubed, and the square root twice extracted, to get the next succeeding number in the Table. In such a Table the differences are less, which makes the numbers approximate faster. The index $\frac{2}{3}$ or rather $\frac{2}{10}$ would be exceeding good, but for the labour of calculating the Table. In $\frac{2}{10}$ the cube of every number must be cubed, and the square root of the 5th root taken, which would be an immense labour. But when such a Table will serve for all numbers; and will converge 6 or 7 times as fast as the square roots.

PROB. II.

To find the relation between the indefinite root of any number, and its logarithm.

The indefinite root of any number is $1 +$ some very small quantity, as $1 + x$; and if m be that root, then $1 + x^m$ may represent any number, m being indefinitely great. Let N be that number,

O 2

then

then $\overline{1+x^m} = N$, or $N^{\frac{1}{m}} = 1+x$, whence $x = N^{\frac{1}{m}} - 1$. But by the property of the logarithmic curve, if t be the subtangent, then $mt \times N^{\frac{1}{m}} - 1 = \log. N$; where if $t = 1$, it will be Neper's Logarithm; if $t = .43429448$, it is the common logarithm.

Cor. 1. If r be any other indefinite root, $r \times N^{\frac{1}{r}} - 1 = \log. N$.

Cor. 2. $m \times N^{\frac{1}{m}} - 1 = r \times N^{\frac{1}{r}} - 1$; for they are both equal to $\frac{\log. N}{t}$.

Cor. 3. If P be any other number, then $N^{\frac{1}{m}} - 1 : P^{\frac{1}{m}} - 1 :: \log. N : \log. P$.

Cor. 4. $m \times N^{\frac{1}{m}} - 1 : r \times P^{\frac{1}{r}} - 1 :: \log. N : \log. P$.

Cor. 5. $\log. N = \frac{N^{\frac{1}{m}} - 1}{10^{\frac{1}{m}} - 1}$. This follows from

Cor. 3. by putting $P = 10$, and its $\log. = 1$.

PROB. III.

To make logarithms by the indefinite roots of the numbers.

From the last Prob. or from Cor. 3. the logarithm of a number N may be found, if its indefinite

...ite root can but be extracted, which indeed is all
...difficulty.

To find the log. N.

Involve 2 to some power large enough for the
...ex, as suppose $2^{23} = 8388608 = m$, which is to
...made use of for all numbers N. Then $mt =$
 $838608 \times .43429448 = 3643126$. Then extract
...square root of N 23 times, which will be

$\frac{1}{m}$, then $mt \times N^{\frac{1}{m}} - 1 = \log. N$, for the com-
...log.

E X A M.

To find the log. 2.

Here $N=2$, and $N^{\frac{1}{m}} = 1.0000000826295879$;

$N^{\frac{1}{m}} - 1 = .07826295879$ (07 denoting 7 cy-

...), then $mt \times N^{\frac{1}{m}} - 1 = 3643126 \times$
 $.07826295879 = .3010300$ for the log. 2.

Or thus.

Let $1 + x^{8388608} = N = 2$. Then by the me-
...of extracting the roots of adfectad equations,
...ing the 3 first terms only, we shall have $1 +$
 $838608x + \frac{8388608 \times 8388607}{2} x^2 \&c. = 2$, or

$4194303\frac{1}{2}xx = \frac{1}{8388608}$, whence $x =$

$\frac{284}{3+e} = .078$; and repeating the operation once

...vice, at last $x = .07826295879$; $= N^{\frac{1}{m}} - 1$,
 $mtx = .3010300$, as before, for the log. 2.

...involving the quantities here is extremely diffi-
cult;

O 3

cult; and besides, it does not answer in large numbers, but diverges.

Or thus.

Extract the square root of 10, for 23 extractions

then $10^{\frac{1}{m}} = 1.00000027448957$, and $10^{\frac{1}{m}} - 1 = .0627448954$; then by Cor. 5. Prob. 11. log. N

$$\frac{N^{\frac{1}{m}} - 1}{.0627448954}.$$

E X A M.

To find the log. 2.

Here $N = 2$, and $N^{\frac{1}{m}} - 1 = .07826295879$ therefore $\frac{.07826295879}{.0627448954} = .3010300$ for the log.

If you chuse to have 10000000 &c. for the index m , you must extract the 5th root, and then the square root of that, which gives the 10th root. Again, extract the 5th root of this, and then the square root, for the 100th root; and proceed thus as often as you have cyphers in the index, which here requires 7 repetitions; and the last root will be $1 + x$. Or else extract the 10th root at once by Rule III. page 269, *Algebra*; which is the quickest way.

It is evident all these methods are impractical from the great labour required in extractions and involutions. Yet as difficult as it is, the method by continually extracting the square root, was made use of, at the first invention of logarithms. But, however, these methods shew us very clearly the nature and constitution of logarithms, and the dependence they have on the numbers themselves.

A R T. VIII.

The Interpolation of Quantities.

P R O B. I.

To investigate a general rule for the interpolation of quantities.

LET $m, n, p, q, \&c.$ be a series of numbers, and $a, b, c, d, \&c.$ a series of quantities, among which another is to be interpolated. Let x be the number in the first series, and y the number to be interpolated in the second series.

Assume the powers of x , affected with unknown coefficients, to represent y ; thus, let $y = g + bx + kxx + lx^3 \&c.$ then to determine $g, b, k, l, \&c.$ we must take

$$g + bm + km^2 + lm^3 \&c. = a.$$

$$g + bn + kn^2 + ln^3 \&c. = b.$$

$$g + bp + kp^2 + lp^3 \&c. = c.$$

$$g + bq + kq^2 + lq^3 \&c. = d, \&c.$$

Here having as many equations, as unknown coefficients, g, b, k, l ; these coefficients will be determined. Hence for 4 quantities we get,

$$\frac{a \times q - p \cdot p - n \cdot q - u - b \times q - p \cdot q - m \cdot p - n + c \times n - m}{q - m \cdot q - n - d \times p - m \cdot p - n \cdot n - m}$$

$$= \frac{p - m \cdot n - m \cdot q - p \cdot p - n \cdot q - m \cdot n - q}{a \times p - n - b \times p - m + c \times n - m - l \times m + n + p}$$

$$= \frac{p - n \cdot p - m \cdot n - m}{b - a - l \times mm + nn + mn - k \times m + n}$$

$$= \frac{b - a}{n - m} - l \times mm + nn + mn - k \times m + n.$$

$$g = \frac{an - bm}{n - m} + lmn \times n + m + kmn.$$

O 4

Therefore

Therefore all the numbers and quantities m, n, p, q , and a, b, c, d , being known, g, b, k, l , will be known.

Then assume any value for x , according to the place it is to have in the series m, n, p, q ; and the correspondent value of y will be known, in the like place, among a, b, c, d ; that is, $y = g + bx + kxx + lx^3$.

Cor. If only 3 quantities are concerned, as m, n, p , and a, b, c . And $y = g + bx + kxx$; then

$$k = \frac{a.p - n - b.p - m + c.n - m}{p - n . p - m . n - m}$$

$$b = \frac{b - a}{n - m} - k \times \overline{m + n}$$

$$g = \frac{an - bm}{n - m} + kmn.$$

S C H O L.

Some other methods of interpolation may be seen in my Treatise of the Differential Method.

P R O B. II.

To find a rule for the interpolation of a quantity, among 4 quantities, at equal distances.

Suppose the numbers (in the last Prob.) m, n, p, q , to be 0, 1, 2, 3. Then by substituting numbers for these letters, we shall have

$$l = \frac{2a - 6b + 6c - 2d}{-12} = \frac{d - a + 3 \times \overline{b - c}}{6}$$

$$k = \frac{a - 2b + c}{2} - 3l$$

$$b = b - a - l - k$$

$$g = a.$$

All these methods are the same with M. De la
 il's, but are very laborious.

Otherwise thus.

Suppose as before a, b, c, d , are equidistant,
 and that x lies between b and c . By Prop. V. *Dif-*
ferential Method, $y = A + xB + x \cdot \frac{x-1}{2} C + x \cdot$

$\frac{-1}{2} \cdot \frac{x-2}{3} D$, or putting $1+v$ for x , and $a,$

d', d'' , for A, B, C, D ; then $y = a + 1+v \cdot d' +$
 $\frac{1+v \cdot v}{2} d'' + \frac{1+v \cdot v \cdot v-1}{6} d'''$. But $b+c-b \cdot v$ is

the value of y by the mean motion, and $b=a+d'$,
 and $c-b=d'+d''$, then this value of y is $a+1+v \cdot$
 $+vd''$; this subtracted from the true value of y ,
 gives $\frac{1+v \cdot v}{2} d'' - vd'' + \frac{1+v \cdot v \cdot v-1}{6} d''' = \frac{v \cdot v-1}{2}$

$+ \frac{v \cdot v-1 \cdot 1+v}{6} d'''$ for the correction =

$+ \frac{1+v}{3} d''' \times \frac{v \cdot v-1}{2} = \frac{v \cdot v-1}{2} \times$

$d'' - \frac{1+v}{3} d'''$.

But (Cor. 1. Prop. II. Diff. Method).

$$b = a + d'$$

$$c = a + 2d' + d''$$

$$d = a + 3d' + 3d'' + d''' \text{, whence}$$

$$b+c = 2a + 3d' + d''$$

$$+d = 2a + 3d' + 3d'' + d''' \text{, subtracted .}$$

$$+c-a+d = -2d' - d'' \text{,}$$

$$\text{and } \frac{v \cdot 1-v}{2} \times \frac{b+c-a+d}{2} = \frac{v \cdot 1-v}{2} \times -d' - \frac{d''}{2} \text{,}$$

and this is extremely near the correction required,
 and

and when $v = \frac{1}{2}$, it is exactly so. Therefore

M = value of y by mean motion, $M + \frac{v \times 1 - v}{2}$

$\times \frac{b + c - a + d}{2} = y$ required, where $M = b$
 $c - b.v.$

Cor. 1. If there be only three equidistant quantities
 and d wanting; then $y = M + \frac{v.1-v}{2} \times \frac{b+c}{2}$

Cor. 2. If there be only three equidistant quantities,
 and a wanting; then $y = M + \frac{v.1-v}{2}$

$\frac{b + c - d}{2}$.

E X A M.

Four longitudes of the moon being given, as follows; to find it at $16^h 22' 16''$. - July 16, 1769

July 16. Noon $11^h 29^m 29^s 34'' = a$

Midnight $0^h 6^m 40^s 25'' = b$

17 Noon $0^h 13^m 47^s 48'' = c$ $7^h 7' 23''$

Midnight $0^h 20^m 51^s 27'' = d$

Here $a = -30^m 26''$, then $c - b = 7^h 7' 23''$, and
 $v = 4^h 22' 16''$, and $1 - v = 7^h 37' 44''$; and

$M = 0^h 6^m 40^s 25'' + \frac{7^h 7' 23'' \times 4^h 22' 16''}{12}$

$0^h 9^m 16^s 6''$.

$b + c = 20^m 28' 13''$

sub. $a + d = 20 \quad 21 \quad 1$

$+ \quad 07 \quad 12$

$v.1-v. = \frac{4 \quad 22 \quad 16}{12} \times \frac{7 \quad 37 \quad 44}{12} = .2316$

and $\frac{.2316}{2} \times \frac{7' \quad 12''}{2} = 25''$

Therefore $y = M + 25'' = 0^h 9^m 16^s 31''$,

but the best way is to turn the quantities into decimals.

PROB. III.

Five or more quantities a, b, c, d, e , being given at equal distances; to interpolate a quantity between b and c ; or between c and d .

By Prop. V. of the Differential Method, we have

$$= A + xB + x \cdot \frac{x-1}{2} C + x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} D$$

$$+ x \cdot \frac{x-1}{2} \cdot \frac{x-2}{3} \cdot \frac{x-3}{4} E. \text{ Put } 1+v=x,$$

and $d', d'', d''', d'''' = B, C, D, E$. Then $y =$

$$a + 1 + v \cdot d' + \frac{1+v \cdot v}{2} d'' - \frac{1+v \cdot v \cdot 1-v}{6} d''' +$$

$$\frac{1+v \cdot v \cdot 1-v \cdot 2-v}{24} d''', \text{ between } b \text{ and } c.$$

But if y is to stand between c and d , put $2+v$

$$= x, \text{ and then } y = a + 2 + v \cdot d' + \frac{2+v \cdot 1+v}{2} d''$$

$$+ \frac{2+v \cdot 1+v \cdot v}{6} d''' - \frac{2+v \cdot 1+v \cdot v \cdot 1-v}{24} d''''.$$

where d', d'', d''', d'''' , are the first, of the first, second, third, and fourth orders of differences. in 6 quantities, put also $x = 2 + v$.

They that want to see more of this matter, may consult the said Differential Method, where he will find general solutions to all questions of this nature, whether the quantities stand at equal or unequal distances.

A R T. IX.

Of finding the Longitude at Sea.

HAVING in my book of Astronomy given a short sketch of a method for finding the longitude, by the moon's distance from a star; I shall here prosecute the subject further, and compleat that method; and also lay down some other rules of like sort, by which the same thing may be performed. But I do not advise such methods to be commonly practised at sea, for they take too much labour and calculation; and besides we are not to expect that any method by the moon, can be so exact as one could wish, owing to her slow motion from the sun; being only 12 degrees in a day out of 360. If we were blessed with such a satellite as Jupiter, that revolves in less than two days, nothing would be more easy. But as the case stands, we cannot reasonably expect such accuracy, without the most laborious calculations, and an incredible degree of care and exactness in making observations; and even the frequent repeating of them. At this rate some people ought to have nothing else to do a ship-board; but there is seldom occasion there for such supernumerary hands. The right use then of these methods is in time of distress, when a ship, by stress of weather, has lost her way, and cannot pursue her reckoning. In such a case as this, such a method becomes useful; for if the ship's longitude can be found this way to a degree or two, it will generally answer the same purpose, as if they had it to a minute or two.

It would not be difficult to make Tables to facilitate the calculations, as well in this as in other methods,

hods, but that would not be worth the while, Fig. the method is to be used so seldom. And when it happens, a man need not grudge the labour of 4 hours, to set the ship right; which may not open above once in a voyage. To do it often would be slavery, indeed, when the common own methods are sufficient to guide a ship in moderate weather. And therefore the seamen will not thank us for imposing such a task upon them; they will curse us for it, as being a thing as useless and useless, as taking in the spheroidal part of the earth. And yet in the time of danger and distress, when the reckoning is lost, and when a ship is rambling she knows not where; they will be glad of such a rule, to set them right; and this is all the use it can have.

P R O B. I.

Having the apparent distance of the moon from a star, and their zenith distances; to find the true distance.

Let M be the apparent place of the moon, and 31 . that of the star; and m, s , their true places; Mm is the moon's parallax lessened by the refraction, and Ss is the star's refraction, both which are known, for the known altitudes. And let Z be the zenith.

Then in the spherical triangle MZS ; there is given the distance MS , and the sides ZM, ZS , the altitudes; to find the angle Z , by Case 11th of Spherical Trigonometry.

The angle Z being found; in the spherical triangle mZs , there are given the sides Zm, Zs , and the angle Z ; to find the distance ms , by Case 8th of Spherical Trigonometry; and ms will be the true distance.

S C H O L.

Fig.

SCHOL.

31. In Prob. XVII. Sect. VI. Astronomy, I have given a solution to this, by approximation. And in the Appendix, an accurate method by the natural sines; and I believe a shorter method cannot be found.

Note, the difference of the sides MS, ms, can never amount to a degree.

PROB. II.

To find the longitude of a ship at sea.

1. To perform this, we must have Tables showing the right ascension and declination of the sun and moon, to minutes and tenths, every day at noon, made for a certain place as *Greenwich*; and likewise of the principal fixt stars, that lie near the moon's way.

2. At the place proposed, the moon's distance from some known star, must be truly taken with a good instrument; and also their altitudes at the time, and the exact time noted down, which (reduced to degrees) is the sun's westing from the ship.

3. The altitudes must be cleared of refraction and parallax, and the apparent distance reduced to the true distance, by Prob. I.

4. Take the longitude as near as you can by the dead reckoning, and convert it into time, which reckon west from *Greenwich*, because the sun's apparent motion is westward. Add this time to the time of observation, gives the hour (*b*) at *Greenwich*, or the time elapsed since noon at *Greenwich* (rejecting 24 hours, if it exceeds). Reduce it to the decimal of a day, which put $= t$, and the remainder of the day $= v$, and get *tv*.

5. Lay the Tables before you, and take the moon's right ascension for four days; two before

ART. IX. THE LONGITUDE.

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And two after, the time at *Greenwich*, which call *Fig.*
B, C, D. Then subtract the sum of the ex- 31.

emes (*A, D*), from the sum of the means (*B, C*),
 and call the remainder *F*. Then the mean change
 of right ascension, in the time *t*, is $= t \times \overline{C - B}$;

and the true change $= t \times \overline{C - B} + \frac{tv}{4} F = y$.

and the moon's true right ascension $= B + t \times \overline{C - B}$
 $\frac{tvF}{4} = B + y$.

6. In like manner, for the moon's declination;
 take four days declination; two before, and two
 after the time, which call *a, b, c, d*; and these of
 contrary denomination must be marked negative.
 Then subtract the sum of the extremes (*a, d*), from
 the sum of the means (*b, c*), which call *f*. Then
 the declination by mean motion, is $= b + t \times \overline{c - b}$;

and the true declination $= b + t \times \overline{c - b} + \frac{tv}{4} f$.

7. Then having the distance of the moon and
 star, and their polar distances, find the angle at
 the pole, (by Case 11th of Spherical Trigonome-
 try), which is the difference of right ascensions of
 the moon and star. This, with the star's right as-
 cension, gives the moon's present right ascension.

8. The moon's present right ascension, and her
 right ascension at noon (by the Tables), being now
 known; we shall have the right ascension gained
 (*G*), in the time *t*, or *b* hours.

9. Then $\frac{Gb}{y} =$ sun's westing from *Greenwich*

hours; from this subtract the time at the ship;
 the difference, turned into degrees, is the longitude
 of the ship.

10. If

Fig. 10. If more exactness be required, either take the last result for a new operation, or else the nearest round number, for the ease of calculation; and find t and b anew; and then repeat the operation. But if the longitude was near the truth, $\frac{1}{4}t\omega F$ and $\frac{1}{4}t\omega f$, may remain the same, as also g and b . And when most of the quantities remain the same, a new operation will soon be done.

30. 11. The most expeditious way of working, is to use decimals for all the quantities. And to give a true notion of the work, always draw a figure. Here $S\odot$ is the time at the ship, $G\odot$ the time at Greenwich, and GS the ship's longitude.

This method may be equally applied to the sun as well as to a star; but with something more trouble. For the change of his right ascension and declination must also be found.

Note, the process, Art. 5 and 6, is founded on Prob. II, of the last article.

II. Otherwise thus.

Here we must have Tables of longitudes and latitudes of the moon and stars. And instead of the right ascension and declination of the sun, moon and stars; make use of their longitudes and latitudes; and work by the very same rule as before. Thus Art. 5. gives the moon's true longitude, and Art. 6. her true latitude. And Art. 7. finds the diff. longitude of the moon and star, and consequently the moon's present longitude. By Art. 8. is had the moon's longitude gained in the time. And so on.

It is worth observing, that the very same rule will answer for the Tables of longitude and latitude as well as for those of the right ascension and declination, *mutatis mutandis*. And it is the peculiar advantage of this method, that the problem may be solved either way; and therefore it may be done

by some of the common almanacks, provided their Fig.
 numbers be true and correct.

30.

SCHOL. I.

When the moon has little or no declination, one
 need not find the quantity F nor f , nor the correction.

In addition and subtraction, the signs of the
 quantities must be carefully observed. And there-
 fore when a greater quantity is to be subtracted
 from a lesser; the remainder must be set down ne-
 gative, as may happen in Art. 5 and 6. Also in
 Art. 9. when the time at the ship is greater; the re-
 mainder is negative, and the longitude east.

When the ship lies east of *Greenwich*, it may do
 as well to reckon that distance negative.

Note, most of the work will be best done by lo-
 garithms.

III. Otherwise thus.

1. From the assumed longitude and given time
 of observation, having got the moon's right ascen-
 sion, by Art. 5. from the Tables;

And likewise having got her right ascension by
 calculation, by Art. 7. compare them together. If
 they are the same, you may be sure your longitude
 is right: if they differ, note the error.

2. Make a nearer supposition for the longitude,
 as you may easily do. Then having found the
 moon's right ascension twice (by Art. 5. and 7.) as
 before; if they differ, note the error again.

Then by the method of trial and error, the true
 longitude of the ship will be found.

And the same thing may be done, by making
 use of the moon's longitude (instead of her right
 ascension), according to Rule II.

Note, one may easily know, as soon as the first
 observation is over, by comparing the right ascen-
 sions, &c. whether the first longitude be taken too

P

great

Fig. great or too little; and take the next accordingly.
30. This Rule comes very near.

S C H O L. 2.

After the mean proportional part of the declination is found; if you would not take the trouble of finding F or f ; take $\frac{1}{3}$ the degrees of declination; and add (always) so many minutes (z) to the declination; and $\frac{1}{3}$ of that to or from the (mean) right ascension. Or more exactly, for 0, 3, 6, 9 hours before or after noon, take 0, $\frac{4}{10}$, $\frac{7}{10}$, or $\frac{9}{10}$ of the greatest correction (z), to be added (always) to the mean declination. When the declination decreases, add; when it increases, subtract the correction to or from the right ascension.

IV. Or thus.

1. From the supposed longitude, and given time of observation, proceed by the first Rule, to get the moon's right ascension (m) by the Tables; and likewise her right ascension (n) by calculation; and find the angle at the moon. Then,

2. Put $\tau = \cotan \angle$ at the moon.

$s = \text{fine. moon's polar distance.}$

$q = c - b$; $p = C - B$, then

$\frac{6 \times n - m}{p - \frac{\tau}{s} q} = \text{correction of the longitude}$

tude of the ship in degrees; to be added or subtracted, according as the sign directs.

Note, p and q are in degrees, but $n - m$ is in minutes; and τ is negative, when the angle is greater than 90 degrees.

For let $t + x$ be the true time, x being the increment of t , then $qx = \text{variation of declination or of the polar distance.}$ And (by Cotes, Theorem 2.)

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23, pa. 17.) $s : r :: qx : \frac{\tau}{s} qx = \text{variation of right}$ Fig. 30.

ascension. Then $m + px = n + \frac{\tau}{s} qx$, and $px -$

$\frac{\tau}{s} qx = n - m$ in degrees; or $px - \frac{\tau}{s} qx = \frac{1}{60} \times$

$n - m$ ($n - m$ being minutes); therefore $x =$

$\frac{1}{60} \times \frac{n - m}{p - \frac{\tau}{s} q}$, the increment of time in days, and

$p - \frac{\tau}{s} q$

$\frac{1}{60} \times \frac{n - m}{p - \frac{\tau}{s} q} = \text{incr. time in hours, and } \frac{24 \times 15}{60}$

$\times \frac{n - m}{p - \frac{\tau}{s} q}$ or $\frac{6 \times n - m}{p - \frac{\tau}{s} q} = \text{increment of the de-}$

grees of longitude.

This is easily wrought by logarithms; and this

correction will come very near, when the supposed

longitude is within a few degrees of the truth.

This is as easily done by making use of the

moon's longitude, instead of her right ascension.

But then the angle at the moon must be found in

the triangle whose two sides are the co-latitudes of

the moon and star, instead of the co-declinations.

EXAM. I.

On April 4th, 1767, at 4^h 28^m $\frac{1}{3}$ apparent time,

the distance of the moon from the sun was observed to

be 74° 16', and altitude of the sun 22° 18', of the

moon 80° 53'. The longitude by the reckoning 17 46

(= 1^h 11^m in time.) To find the ship's true lon-

gitude.

Here $Ss = 3'$, $Mm = 9'$. Therefore there is 31.

given, ZM , ZS , MS , and Zm , Zs ; then by Prob.

ms is found = 74° 12'.

P 2

Preparation.

OF FINDING

Preparation.

By Rule I.

30.

Long time $1^h 11^m$
 time of observat. $4 \ 28^{\frac{1}{3}}$

$$5 \ 39^{\frac{1}{3}} = 5.655 = b$$

$$\frac{5.655}{24} = .2356 = t \quad \left. \begin{array}{l} \\ .7644 = v \end{array} \right\} tv = .18,$$

$$A = 71^{\circ}.050$$

$$a = 26^{\circ}.483N$$

$$B = 85.533$$

$$b = 26.783$$

$$C = 99.483$$

$$c = 25.650$$

$$D = 112.706$$

$$d = 23.300$$

$$B+C = 185.016$$

$$b+c = 52.433$$

$$A+D = 183.766$$

$$a+d = 49.783$$

$$F = 1.250$$

$$f = 2.650$$

$$C-B = 13.950$$

$$c-b = - 1.133$$

$$C-B \times t = 3.286$$

$$t \times c-b = - .266$$

$$\frac{1}{4} tv F = .056$$

$$\frac{1}{4} tv f = .119$$

$$y = 3.342$$

$$b = 26.783$$

$$B+y = 88.875$$

$$D's \text{ true declin. } 26.636$$

the D 's right ascension.

$$D's \text{ pol. dist. } = 63.304$$

$$63^{\circ} 22'$$

$\odot$'s change of declin. in a day

is $22'.8$

$$\text{and } .235 \times 22.8 = 5.3$$

$$\odot \text{ decl. Ap. 4th } 5 \ 42.8$$

$$\odot \text{ present decl. } 5 \ 48.1$$

Change $\odot$ r. asc. in a day, $54^{\circ}.5$

$$\text{and } .2354 \times 54.5 = 12.8$$

$$\odot \text{ r. ascens. Ap. 4. } 13 \ 19.5$$

$$\odot \text{ present r. asc. } 13 \ 32.5$$

Operation.

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Fig.

Operation.

Base, $\odot$ dist. from $\triangleright$ $74^{\circ} 12$

32.

S. $\triangleright$ polar dist. $63\ 22$, co. ar. 0.0487142
S. $\odot$ polar dist. $84\ 12$, co. ar. 0.0022290

diff. sides $20\ 50$

+ base $95\ 02$

half $47\ 31$, S.

9.8677466

— diff. sides $26\ 41$, S.

9.6523035

$2)19.5709933$

S. $37\ 36.3$

9.7854966

2

$\angle P = 75\ 12.6$ diff. right ascen.

$13\ 32.3$ $\odot$ r. ascension

$88\ 45$ $\triangleright$ s r. ascen. now

$85\ 32$ $\triangleright$ r. af. at noon

G = $3\ 13$ r. af. gained

or 3.216

then $\frac{3.216 \times 5.655}{3.342} = 5^h.44 = 5^h\ 26^m$

And $5^h\ 26^m$ $\odot$ west from Greenwich

— $4\ 28$ time at the ship

$\odot\ 58 = 14\frac{1}{2}$ deg. the longitude west.

The same Ex. by Rule II.

The quantities *ms*, *b*, *t*, *v* and *tv* remain the same; then,

P 3

Preparation.

OF FINDING

Preparation.

$A = 73.053$	$a = 4.130N$
$B = 86.006$	$b = 3.364$
$C = 98.556$	$c = 2.454$
$D = 110.782$	$d = 1.454$
$B+C = 184.562$	$b+c = 5.818$
$A+D = 183.835$	$a+d = 5.584$
$F = .727$	$f = .234$
$C-B = 12.550$	$c-b = -.910$
$C-B \times t = 2.957$	$t \times c-b = -.215$
$\frac{1}{4}tcf = 0.033$	$\frac{1}{4}tcf = .105$
$y = 2.990$	$b = 3.364$
$B+y = 88.996$	$\text{D's true lat. } 3.256$
D's longitude	$\text{D's co. lat. } 86.744$
	$\text{or } 86.44\frac{1}{2}$

Operation.

32. In the quadrantal triangle MPS, where P is the pole of the ecliptic, and PS the sun's polar distance $= 90^\circ$; MP the D's colat. $86^\circ 44\frac{1}{2}'$; MS the distance of the sun and moon $74^\circ 12'$. To find the $\angle$ P.

$$\text{Cof. } M \text{ or } S. 86.44\frac{1}{2} \quad 9.9991473$$

rad. 10.

$$\text{cof. } 74.12 \quad 9.4350161$$

$$\text{cof. } nS, \text{ or cof. } P, 74^\circ 10' \quad 9.4358688$$

$$\text{then dif. lon. of } \odot \text{ and } \text{D} = 74^\circ 10'$$

$$\odot \text{'s long. } 14.42\frac{1}{2}$$

$$\text{D longit. now } 88.52\frac{1}{2}$$

$$\text{D long. at noon } 86.00\frac{1}{2}$$

$$\text{longit. gained } 2.52 = 2.866$$

$$\text{then } \frac{Gb}{y} = 5^h.42 = 5^h 25^m \odot \text{ west from Greenwich.}$$

$$- 4.28 \text{ time at the ship.}$$

$$\odot 57 = 14\frac{1}{2} \text{ the long. west}$$

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If this had been wrought by mean motion, we should have, the moon's right ascension $88^{\circ} 49'$ Fig.
the moon's declination $26^{\circ} 31'$ 33.
and the longitude 16 degrees.

EXAM. 2.

Sept. 5th, 1767, at $15^h 7^m \frac{1}{3}$ apparent time, the corrected distance of the moon from the star α pegasi was $44^{\circ} 0' \frac{1}{3}$; the longitude by the reckoning $64^{\circ} 32' E = 4^h 18^m$ time). To find the true longitude.

By RULE I.

Preparation.

$$\begin{aligned}
 & 4^h 18^m \text{ longitude time} \\
 & - 15 \ 7 \frac{1}{3} \text{ time at the ship.} \\
 & - 10 \ 49 \frac{1}{3} = b, \text{ hour at Greenwich.} \\
 & \text{or } 10.822 = b, \\
 & \text{and } \frac{10.822}{24} = .4509 = t \} tv = .2476 \\
 & \qquad \qquad .5491 = v \}
 \end{aligned}$$

$$\begin{aligned}
 A &= 290^{\circ} 16' & a &= 23^{\circ} 13' S \\
 B &= 305 \ 5 & b &= 19 \ 19 \\
 C &= 319 \ 31 & c &= 14 \ 7 \\
 D &= 333 \ 35 & d &= 7 \ 58 \\
 B+C &= 624 \ 36 & b+c &= 33 \ 26 \\
 A+D &= 623 \ 51 & a+d &= 31 \ 11 \\
 & \qquad 0 \ 45 & f &= + \ 2 \ 15 \\
 \text{or } F &= .75 & \text{or } f &= 2.25 \\
 C-B &= 14 \ 26 & c-b &= - \ 5 \ 12 = -5.2 \\
 \text{or } & 14.433 & c-b \times t &= - \ 2.345 \\
 C-B \times t &= 6.508 & \frac{1}{4} tvf &= .138 \\
 \frac{1}{4} tvF &= .046 & b &= 19.316 \\
 y &= 6.554 & D's \text{ true decl.} & 17.109 \\
 B &= 305.083 & & 17 \ 6.7 \\
 B+y &= 311.637 & D's \text{ polar } & \} \\
 \text{the } D's \text{ right ascens.} & & \text{distance } & \} 107^{\circ} 6.7'
 \end{aligned}$$

P 4

Operation.

Fig.
34.

Operation.

Base, or * dist. from D, $\overset{\circ}{44} \overset{\circ}{0}\frac{1}{2}$

S. D's polar dist. 107 6.7 coar. 0.019662

S. *'s polar dist. 76 2 coar. 0.013033

diff. fides 31 4.7

+ base 75 5

half 37 32.5, S. 9.784858

— dif. fides 6 27.8, S. 9.051449

2) 18.869003

S. 15 46.8

9.434501

2

< P = 31 33.6 diff. right ascen.

343 18 * right ascen.

311 44.4 D's right af. now

305 5 D's r. asc. at noon

G = 6 39.4 D's r. af. gained.

= 6.656

then $\frac{Gb}{y} = 10^h.99 = 10^h 59^m.3$, $\odot$ west from
Greenwich.

— 15 7.3 time at the ship

4 8 = 62° the longi. E

The longitude by mean motion comes out $63^\circ 45'$ *Again, for another operation.*Assume 60 for the longitude of the ship = 4
hours.*Preparation*

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Fig.

34.

Preparation.

$$\begin{array}{r} 4^h 0 \text{ supposed time} \\ - 15 \ 7^{\frac{1}{2}} \text{ time at ship} \\ \hline \end{array}$$

$$11 \ 7^{\frac{1}{2}} = 11.122 = b, \text{ the time at Greenwich.}$$

$$\begin{array}{r} 11.122 \\ 24 \end{array} = .4634 = t \quad \begin{array}{r} .5366 = v \end{array} \quad tv = .2486$$

$$C - B = 6.688 \quad t \times c - b = - 2.409$$

$$B = 305.083 \quad \frac{1}{4}tvf = .138$$

$$\frac{1}{4}vF = .046 \quad b = 19.317$$

$$B + y = 311.817 \quad 17.046$$

$$\text{D's right ascens.} \quad \text{the D's declination.}$$

$$\text{and } y = 6.734, \quad \text{D's polar dist.} = 72^\circ 57'$$

Operation.

$$\text{base, or D dist. from } * \quad 44 \quad 0.3$$

$$\text{S. D's polar dist.} \quad 107 \quad 3, \quad \text{coar. } 0.0195197$$

$$\text{S. } * \text{ polar dist.} \quad 76 \quad 2, \quad \text{coar. } 0.0130330$$

$$\text{diff. sides} \quad 31 \quad 1$$

$$+ \text{ base} \quad 75 \quad 1.3$$

$$\text{half} \quad 37 \quad 30.6, \text{ S.} \quad 9.7845569$$

$$- \text{ diff. sides} \quad 6 \quad 29.6, \text{ S.} \quad 9.0534890$$

$$2) 18.8705986$$

$$\text{S.} \quad 15 \quad 48^{\frac{1}{2}} \quad 9.4352993$$

2

$$\angle P = 31 \ 37 \text{ dif. right ascens.}$$

$$343 \ 18 \ * \text{ right ascen.}$$

$$311 \ 41 \ \text{D right ascen. now}$$

$$305 \ 5 \ \text{D right af. at noon}$$

$$G = 6 \ 36 \ \text{D right af. gained}$$

$$\text{or} \quad 6.6$$

then

Fig. then $\frac{Gb}{y} = 10^h.9$ @ west from Greenwich.

$$34. \quad - 15^h 7.3^m = - 15.12 \text{ time at the ship.}$$

4.22 the long. in hours E.

$4.22 \times 15 = 63^{\circ}.3 = 63^{\circ} 18'$ the longitude E. degrees.

Calculation of the same for $62^{\circ} 40' (= 4^h 10^m$

Preparation.

$$\begin{array}{r} h \quad m \\ 4 \quad 10.6 \text{ supposed time} \\ - 15 \quad 7.3 \text{ time at the ship} \\ \hline \end{array}$$

$$10 \quad 56.7 = 10.945 = b, \text{ hour at Greenwich}$$

$$\frac{10.945}{24} = .456 = t, \text{ } tw = .248$$

$$\begin{array}{r} t \times C - B = 6.581 \quad t \times c - b = - 2.37 \\ \frac{1}{4} tw F = .047 \quad \frac{1}{4} tw f = .14 \end{array}$$

$$y = 6.628 \quad b = 19.317$$

$$B = 305.083 \quad D's \text{ declin. } 17.087 = 17^{\circ} 5'$$

$$B + y = 311.711 \quad D's \text{ pol. dist. } 107^{\circ} 5'$$

$$= 311.42.6$$

the D's right ascen.

Operation.

$$\text{Base, } D \text{ dist. from } * \quad 44 \quad 0.3$$

$$\text{S. } D's \text{ polar dist. } 107 \quad 5.2 \quad \text{coar. } 0.019603$$

$$\text{S. } * \text{ polar dist. } 76 \quad 2 \quad \text{coar. } 0.013033$$

$$\text{diff. sides } 31 \quad 3.2$$

$$+ \text{ base } 75 \quad 3.5$$

$$\text{half } 37 \quad 31.7 \text{ S. } 9.784721$$

$$- \text{ diff. sides } 6 \quad 28.5 \text{ S. } 9.052192$$

$$2) 18.869550$$

$$9.434775$$

$$\text{S. } 15 \quad 47.5$$

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S. 15 47.5
2

Fig.
34.

$$\begin{array}{r} \angle P = \begin{array}{r} 31 \quad 35 \\ 343 \quad 18 \end{array} \text{ diff. right ascen.} \\ \quad \quad \quad \begin{array}{r} 311 \quad 43 \\ 305 \quad 5 \end{array} \text{ * right ascen.} \\ \quad \quad \quad \begin{array}{r} 311 \quad 43 \\ 305 \quad 5 \end{array} \text{ D's right af. now} \\ \quad \quad \quad \begin{array}{r} 305 \quad 5 \\ 6 \quad 38 \end{array} \text{ D's right af. at noon} \\ G = \begin{array}{r} 6 \quad 38 \\ 6.633 \end{array} \text{ right ascen. gained} \\ \text{or} = \begin{array}{r} 6 \quad 38 \\ 6.633 \end{array} \end{array}$$

$$\text{then } \frac{Gb}{y} = 10^h.95 = 10 \ 57$$

$$\begin{array}{r} - \quad 15 \quad 7.3 \\ - \quad 4 \quad 10.3 \end{array} = 62^\circ \ 35' \text{ the longitude E.}$$

The same Exam. by Rule II.

Preparation.

Having found as before $t = .451$, $w = .2476$, 35°
 $b = 10.882$, from the Tables of longitude and
itude, we shall have

$A = 288.566$	$a = - 1.043 \ S$
$B = 302.847$	$b = + 0.242 \ N$
$C = 317.558$	$c = 1.548$
$D = 332.613$	$d = 2.772$
$B+C = 620.405$	$b+c = 1.790$
$A+D = 621.179$	$a+d = 1.729$
$F = - 0.774$	$f = .061$
$C-B = 14.711$	$c-b = 1.306$
	$c-b \times t = .588$
$C-B \times t = 6.634$	$\frac{1}{4} w f = .004$
$\frac{1}{4} w F = - .048$	$b = .242$
$y = 6.586$	D's true lat. $.834$
$B+y = 309.433$	D's polar dist. 89.166
the D's longitude.	or $89^\circ \ 10'$

Operation:

OF FINDING

Operation.

Base, or * dist. from D — $\begin{array}{r} 44 \\ 0.3 \end{array}$

S. D's co-lat $\begin{array}{r} 89 \\ 10 \end{array}$ coar. 0.00004

S. *'s co-lat. $\begin{array}{r} 70 \\ 35.3 \end{array}$ coar. 0.02544

diff. sides $\begin{array}{r} 18 \\ 34.7 \end{array}$

+ base $\begin{array}{r} 62 \\ 35 \end{array}$

S. half $\begin{array}{r} 31 \\ 17.5 \end{array}$ S. 9.71549

— diff. sides, S. $\begin{array}{r} 12 \\ 42.8 \end{array}$ S. 9.34256

2) 19.08335

S. $\begin{array}{r} 20 \\ 22.45 \end{array}$

2

9.54176

< P = $\begin{array}{r} 40 \\ 44.9 \end{array}$ diff. lon. D and

$\begin{array}{r} 350 \\ 14.5 \end{array}$ long. *

$\begin{array}{r} 309^\circ \\ 29'.6 \end{array}$ D's long. now

$\begin{array}{r} 302 \\ 50.8 \end{array}$ D's long. at noon

$\begin{array}{r} 6.38.8 \end{array}$ longitude gained

or $\begin{array}{r} 6.646 \end{array}$ = G,

then $\frac{Gb}{y} = 10^h.92$

— time at the ship — $\begin{array}{r} 15.12 \end{array}$

$\begin{array}{r} 4.20 \end{array}$

and $4.20 \times 15 = 63^\circ$ the long. east

Otherwise thus, by Rule III.

1. Suppose (as in the first calculation) that the longitude is $64^\circ 32'$, then

D's right ascension is $311^\circ 38'.2$ by the Tables

and her right ascen. $311^\circ 44.4$ by calculation

difference + $\begin{array}{r} 6.2 \end{array}$ long. too great

2. Suppose

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Suppose (as in the second calculation) that the Fig. longitude is $60^{\circ} 00'$, then

's right ascension is $311^{\circ} 49'$ by the Tables, 34.
and her right ascension $311 \ 41$ by calculation.

difference — 8 long. too little.

Then by the rule of false, $\frac{4^{\circ}.533 \times 6.2}{6.2 + 8} = 1^{\circ}.98$

$1^{\circ} 59'$ the correction, and $64^{\circ} 32' - 1^{\circ} 59' = 62^{\circ} 33'$ for the longitude, where $4^{\circ}.533 = 64 \ 32$
60.

EXAM. 3.

Jan. 13th, 1767, at $10^h 27^m \frac{1}{4}$, the distance of the from regulus, when corrected, was $46^{\circ} 32' \frac{1}{2}$. To 35.
the longitude.

As here is no longitude given, we shall assume longitude, for the first trial.

Preparation.

long. in time $0^h \ 0^m$

of observ. $10 \ 27 \frac{1}{4}$

$$10 \ 27 \frac{1}{4} = 10.454 = b.$$

$$\text{and } \frac{10.454}{24} = .4356 = t \quad .5644 = v \quad \} \quad tv = .2458$$

$$A = \begin{array}{r} 83 \\ 11 \end{array}$$

$$B = \begin{array}{r} 96 \\ 59 \end{array}$$

$$C = \begin{array}{r} 110 \\ 13 \end{array}$$

$$D = \begin{array}{r} 122 \end{array}$$

$$a = \begin{array}{r} 27 \\ 7 \text{ N} \end{array}$$

$$b = \begin{array}{r} 26 \\ 15 \end{array}$$

$$c = \begin{array}{r} 24 \\ 7 \end{array}$$

$$d = \begin{array}{r} 20 \\ 56 \end{array}$$

$$B+C = \begin{array}{r} 207 \\ 12 \end{array}$$

$$A+D = \begin{array}{r} 205 \\ 59 \end{array}$$

$$b+c = \begin{array}{r} 50 \\ 22 \end{array}$$

$$a+d = \begin{array}{r} 48 \\ 3 \end{array}$$

$$F = \begin{array}{r} 1 \\ 13 \end{array}$$

$$\text{or } F = \begin{array}{r} 1.217 \end{array}$$

$$C-B = \begin{array}{r} 13 \\ 14 \end{array}$$

$$\text{or } = \begin{array}{r} 13.233 \end{array}$$

$$f = \begin{array}{r} 2 \\ 19 \end{array}$$

$$\text{or } f = \begin{array}{r} 2.317 \end{array}$$

$$c-b = \begin{array}{r} 2 \\ 8 \end{array}$$

$$\text{or } = \begin{array}{r} 2.133 \end{array}$$

IX

Fig. $t \times \overline{C-B} = 5.751$ $t \times \overline{c-b} = -$.927
 35. $\frac{1}{4} t v F = .075$ $\frac{1}{4} t v f = .142$
 $y = 5.826$ $b = 26.250$
 $B = 96.983$ 25.403
 102.809 or 25 28
 or 102 49 D's declination.
 the D right ascension. D pol. dist. $\overline{64 \ 32}$

Operation.

36. Base, D dist. from *, $\overline{46 \ 32 \frac{1}{2}}$
 S. D's polar dist. $\overline{64 \ 32}$, coar. 0.04438
 S. * polar dist. $\overline{76 \ 54}$, coar. 0.01145
 diff. sides $\overline{12 \ 22}$
 + base $\overline{58 \ 54 \frac{1}{2}}$
 half $\overline{29 \ 27}$ S. 9.69166
 — diff. sides $\overline{17 \ 5}$ S. 9.46799
 2) 19.21550
 S. $\overline{23 \ 54 \frac{1}{2}}$ 9.60775
 2

< P = $\overline{47 \ 49}$ dif. right ascension
 $\overline{148 \ 59}$ * right ascension
 $\overline{101 \ 10}$ D right ascens. now
 $\overline{96 \ 59}$ D right af. at noon

G = $\overline{4 \ 11}$ right ascens. gained
 or $\overline{4.183} = G$

then $\frac{Gb}{y} = 7^h.505 = 7^h \ 30^m$

— $\overline{10 \ 27}$
 — $\overline{2 \ 57}$

and $2^h \ 57^m = 44^\circ \frac{1}{4}$ longitude E.

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Then for a second trial, take the long. 44° ; or Fig
her 40° E. here now t &c. being found, and the ^{36.}
eration performed, the true longitude will come
 $43\frac{1}{4}^{\circ}$ E.

If only the mean motion had been used, the lon-
gitude would have come out $43^{\circ} 0'$.

Otherwise solved, by Rule IV.

In the foregoing Example, suppose the longitude
E. then

$$\text{long. } 40^{\circ} \text{ E.} = - \quad 2^h \ 40^m$$

$$\text{time at ship} = 10 \ 27.2$$

$$7 \ 47.2 = 7^h.787$$

$$\text{and } \frac{7.787}{24} = .3244 = t \quad \left. \begin{array}{l} .3244 = t \\ .6756 = v \end{array} \right\} tv = .2192.$$

Preparation.

having found

35.

$$\overline{C-B} = \overset{\circ}{13.233} = p \quad c-b = - \overset{\circ}{2.133} = q$$

$$\text{en } t \times \overline{C-B} = 4.293 \quad t \times c-b = - .691$$

$${}^{\frac{1}{2}}tvF = .066 \quad b = 26.250$$

$$y = 4.359 \quad {}^{\frac{1}{2}}tvf = .127$$

$$B = 96.983 \quad 25.686$$

$$101.342$$

$$\text{or } 25 \ 41$$

$$m = 101^{\circ} \ 20'$$

$$\text{D's declination.}$$

$$\text{the } \text{D's right ascension. } \text{D's polar dist. } 64 \ 19$$

Operation

224

Fig.

36.

OF FINDING

Operation for the $\angle P$.

$$\begin{array}{rcl}
 \text{Base SM, } \triangleright \text{ dist. from } * & \underline{46 \ 32\frac{1}{2}} & \\
 \triangleright \text{ pol. distance } & \underline{64 \ 19} & \text{coar. } 0.0451 \\
 * \text{ pol. dist. } & \underline{76 \ 54} & \text{coar. } 0.0114 \\
 \text{diff. sides} & 12 \ 35 & \\
 + \text{ base} & 59 \ 7\frac{1}{2} & \\
 \text{half} & 29 \ 34 & \text{S. } 9.6932 \\
 \text{diff. sides} & \underline{16 \ 59} & \text{S. } 9.4655 \\
 & & 2) 19.2157 \\
 & & 9.6078
 \end{array}$$

$$\text{S. } 23 \ 54.3$$

$$\begin{array}{rcl}
 \angle P = & \underline{47 \ 48\frac{1}{2}} & \text{diff. right ascen.} \\
 & \underline{148 \ 59\frac{1}{2}} & * \text{ right ascen.} \\
 n = & \underline{101 \ 11} & \triangleright \text{ right ascen.}
 \end{array}$$

Operation for the $\angle M$.

$$\begin{array}{rcl}
 \text{S. SM } (46 \ 32\frac{1}{2}) & \text{coar. } 0.13914 & \\
 \text{S. P } (47 \ 48\frac{1}{2}) & 9.86976 & \\
 \text{S. SP } (76 \ 54) & 9.98855 & \\
 \text{S. M } (96 \ 12) & \underline{9.99745} &
 \end{array}$$

$$\begin{array}{rcl}
 \text{then } n = & 101 \ 11 & \\
 m = & 101 \ 20 &
 \end{array}$$

$$n - m = - \quad 9'$$

$$\tau = - \quad .109 \quad p = 13.233$$

$$s = \quad .901 \quad - \frac{\tau}{s} q = .255$$

$$q = - \quad 2.133$$

$$p = \quad 13.233$$

$$\underline{12.978} = 13 \text{ near}$$

$$\frac{\tau}{s} q = \quad .255$$

$$\begin{array}{rcl}
 - \quad 40 \ 0 & 54 & = - 4.1 = - 4^\circ 6' \\
 - \quad 4 \ 6 & 13 & \\
 \hline
 44 \ 6 \text{ E. long.} & &
 \end{array}$$

But

ART. IX. THE LONGITUDE: 225

But this method requires too much labour; Fig.
Therefore rather work by Rule III. 36.

SCHOL.

Any method of finding the longitude by the
sun is clogged with many difficulties; as,

1. The uncertainty of the sun's declination, when
observation is made for getting the time.
2. The uncertainty of the ship's latitude, at the
time of observation.
3. The difficulty of taking the distance of the
sun exactly from the sun or a star, particularly
the sun.
4. The inaccuracy of the Tables made use of in
the calculations.
5. The danger of mistakes by the numerous cal-
culations to be gone through by any of these me-
thods.

Q

A R T.

A R T. X.

Of Interest.

HERE I shall compute the interest, and amount, of any sums or annuities for any time in general, as well as whole years, at both simple and compound interest; and also their present worths.

Let p = principal or present worth.

a = annuity or yearly rent.

s = amount or sum of all the arrears.

T = whole time of continuance.

b = fractional part of a year.

t = the even years.

R = the sum of 1*l*. and its interest for a year.

r = the interest of 1*l*. for a year.

hence $T = t + b$, and $R = 1 + r$.

P R O P. I.

The principal, time, and rate of interest being given; to find the amount or arrear; at simple interest.

As $1 : r :: p : rp$, the interest of p for a year
And $1 : rp :: T : Trp$, the interest of p for the time T . Therefore $p + Trp$ = whole arrear at the end of the time T ; whence $p + prT$ or $p + prb + prtb$ = whole amount = s .

Cor. 1. pTr = interest of p for the time T .

Cor. 2. $\frac{sTr}{Tr+1}$ = the discount of s for the time

T , whereof p is the rebate.

Cor. 3. In equation of payments, time =
 $\frac{\text{sum of the discounts}}{\text{sum pref. worths} \times r}$

P R O P. II.

Annuity, time, and rate of interest being given; to find the amount or arrear, at the end of the time, at simple interest.

By Cor. 1. Prop. I.

bra = interest of the rent due at t years end.

$1+b.ra$ = interest of the rent due at $t-1$ years end.

$2+b.ra$ = interest of the rent due at $t-2$ years end.

$3+b.ra$ = interest of the rent due at $t-3$ years end.

$4+b.ra$ = interest of the rent due at $t-4$ years end.

and so on till

$t-1+b.ra$ = interest of the rent due at 1 years end

Therefore if a be the rent, the sum will be $bra +$

$1+b.ra + 2+b.ra + 3+b.ra$ &c. to $n-1+b.ra$

$= b + 1+b + 2+b + 3+b \dots t-1+b : \times ra,$

for the whole interest. But the series, consisting of

t terms, its sum is $= \frac{t-1+2b}{2} \times tra,$ to which

add the sum of the rents Ta , then $s = \frac{t.t-1+2bt}{2}$

$\times ra + Ta.$

Cor. 1. $a = \frac{2s}{tt-t+2bt \times r + 2T}$

Cor. 2. $r = \frac{\frac{2s}{a} - 2T}{tt-t+2bt}$

Cor. 3. To find T , put $b = 0$, then $2s = tt - t.ra + 2t$, and $tt + \frac{2-ra}{ra} t = \frac{2s}{ra}$, whence t will be

Q 2 found,

found, which will be something too big, but the next lesser whole number is to be taken. Then to find b , we have $2s - 2ta - 2ba = \frac{tt-t+2bt}{2} \times ra$, and $2bira + 2ba = 2s - 2ta + tra - ttr$, and $b = \frac{\frac{2s}{a} - 2t + tr - ttr}{2tr + 2}$; when this does not answer, put another whole number for t .

P R O P. III.

Having given the annuity, time of continuance, and rate of simple interest; to find the present worth.

By Prop. I. $p + prT = s$, and by Prop. II. $\frac{tt-t+2bt}{2} \times ra + Ta = s$; therefore $p + prT = \frac{tt-t+2bt}{2} \times ra + Ta$, whence $p = \frac{\frac{tt-t+2bt}{2} \times ra + 2Ta}{2rT + 2}$.

$$\text{Cor. 1. } a = \frac{2rT + 2 \times p}{\frac{tt-t+2bt}{2} \times r + 2T}$$

$$\text{Cor. 2. } r = \frac{2Ta - 2p}{2pT + t - tt - 2bt \times a}$$

Cor. 3. To find T , put $b = 0$, then in the equation $p + prt = \frac{tt-t}{2} ra + ta$, t will be found, the nearest whole number. Then proceed as in Cor. 3. of the last Prop. to find b .

S C H O L.

In simple interest, each sum of money is to gain interest after it becomes due. And compound interest, or interest upon interest, supposes that the interest itself shall also gain interest.

P R O P.

PROP. IV.

Having given the principal, rate of interest, and time of continuance; to find the amount, at compound interest.

R = amount of 1*l.* in a year.

$R :: R : R^2$, amount in 2 years.

$R :: R^2 : R^3$, amount in 3 years.

$R^b :: R^t : R^{t+b}$, the amount in $t + b$ years.

Hence $1 : R^{t+b} :: p : pR^{t+b}$ the amount of p in the time $t + b$ or T . Therefore $pR^{t+b} = s$.

Cor. 1. $\frac{R^{t+b} - 1}{R - 1} \times p =$ the compound interest of p for the time T .

Cor. 2. $p = \frac{s}{R^{t+b}}$.

Cor. 3. $\frac{R^{t+b} - 1}{R - 1} \times s =$ discount or rebate of s for the time T .

For the discount is the difference between the

s , and its present worth, $= s - p = s - \frac{s}{R^{t+b}}$

$$\frac{R^{t+b}s - s}{R^{t+b}}$$

PROP. V.

Having given the annuity, time, and rate of interest; to find the money due at the end of that time, at compound interest.

By Prop. V. we shall have

$=$ rent due at the end of T or $t + b$ years without interest.

$=$ rent due at t years end, and its interest till T .

$+b =$ rent due at $t - 1$ years end, and interest till T .

Q 3

aR

$aR^{t+2}+b$ = rent due at $t+2$ years end, and interest till T

$aR^{t+3}+b$ = rent due at $t+3$ years end, and interest till T
and so on, till

$aR^{t+1}+b$ = rent due at 1 year's end, and interest till the end of T years.

Therefore $s = ba + R^b + R^{1+b} + R^{2+b} + R^{3+b} + \dots + R^{t+1}+b : \times$ into a . = $Za + ba$, by substitution

But by Cor. 3. Prop. XXVI. Geom. Proportion the sum of the series $R^b + R^{1+b} + R^{2+b}$ &c. con-

sisting of t terms = $\frac{R^{t+1}-1}{R-1} R^b = Z$: therefore

$$s = ba + Za = ba + \frac{R^{t+1}-1}{r} R^b \times a.$$

$$\text{Cor. 1. } a = \frac{rs}{R^{t+1}+b - R^b + br}.$$

$$\text{Cor. 2. } R^{t+1}+b - R^b + bR - \frac{Rs}{a} = b - \frac{s}{a}$$

From which equation, R may be found, by extracting its root.

Cor. 3. To find T ; put $b=0$, then $aR^{t+1} - Ra - s$, whence t will be found, the nearest whole number. Then by the equation, Cor. 2. b will be found the same way.

P R O P. VI.

Having given the annuity, time, and rate of interest; to find its present worth; at compound interest

By Prop. IV. $s = pR^{t+1}+b$, and by Prop. V. $s =$
 $+ \frac{R^{t+1}+b - R^b}{r} \times a$; therefore $p R^{t+1}+b = ba$

$\frac{R^{t+1}+b - R^b}{r} a$, then $prR^{t+1}+b = bra + \frac{R^{t+1}+b - R^b}{r} a$

$$\times a; \text{ and } p = \frac{bra + R^{t+1}+ba - R^b a}{rR^{t+1}+b}.$$

Cor. 1. $a = \frac{prR^{t+b}}{br + R^{t+b} - R^b}$.

Cor. 2. $pR^{t+b+1} - pR^{t+b} = baR - ba + R^{t+b}a - R^ba$,

By which equation R may be found by extracting the root.

Cor. 3. To find T ; put $b=0$, then $prR^t = R^ta - a$, or $R^t = \frac{a}{a-pr}$, whence t will be found a near whole number, which being had, b will be found, with a deal of trouble, by the equation $prR^t \times R^b = bra + \overline{R^t - 1} \times R^ba$.

Cor. 4. If t be infinite, then $pr = a$, and $p = \frac{a}{r}$,

and $\frac{p}{a} = \frac{1}{r} = \text{number of years purchase}$.

For then $prR^{t+b} = R^{t+b}a$, as all the other quantities are infinitely less, whence $pr = a$.

These propositions and corollaries are best wrought by the help of logarithms; and $1+r$ may be put for R , as there is occasion, and the contrary; and also $t+b$ for T .

S C H O L.

If b be put $= 0$, in these propositions; then will be had all the rules which are commonly laid down, by the writers upon interest.

These last propositions, suppose compound interest to be paid for parts of a year, which admit only of simple interest by law. Whence they are indeed no more than a mathematical speculation; as that method is not practised. Under a year compound interest converts to simple interest.

Fig.

A R T. XI.

The Figures of Sines, Cosines, &c. erected upon all Points of the Arch; their Areas; and Solidities when erected upon the Surface of the Sphere.

P R O P. I.

To find the area of the figure of versed sines, sines, cosines, tangents and secants.

HERE we suppose the arch of the circle, divided into an infinite number of equal parts, upon each of which is erected the sine, cosine, &c. belonging to that arch. To find the area of the curve formed thereby.

37. 1. *For the versed sines.* Let radius $AC = r$, arch $AB = z$, versed sine $AP = v$, sine $BP = y$, cosine $CP = x$, tangent $AT = t$, secant $CT = s$; then $2rv - vv = yy$, and $rv - v\dot{v} = y\dot{y}$. Make $Ab =$ arch AB , and ordinate $bp = Ap = v$. Then the fluxion of the area $Abp = v\dot{z}$, but by the nature of the circle $\dot{z} = \frac{rv}{y}$, and $v\dot{z} = \frac{rv\dot{v}}{y} = \frac{rv\dot{v}}{\sqrt{2rv - vv}}$
 $= \frac{rv\frac{1}{2}\dot{v}}{\sqrt{2r - v}}$. But by Form 10th, Fluxions, the fluent of $\frac{v - \frac{1}{2}\dot{v}}{\sqrt{2r - v}} = \frac{\text{arch } AB}{r} = \phi$. And (Form 11th) fluent of $\frac{v\frac{1}{2}\dot{v}}{\sqrt{2r - v}} = r\phi - \sqrt{2rv - vv} =$
 arch $AB - y$; and fluent of $\frac{rv\frac{1}{2}\dot{v}}{\sqrt{2r - v}}$ or $\frac{rv\dot{v}}{\sqrt{2rv - vv}} =$

ART. XI. FIGURE OF SINES, &c. 233

$r \times \text{arch AB} - \text{fine } y$, for the area of the Fig.
curve Abp .

And when $Ab = \text{quadrant AD}$, the area =
 $\frac{1416rr}{2} - rr$.

And when $Ab = \text{a semi-circle}$, the area =
 $\frac{1416rr}{2} = \text{AFG}$.

2. For the *sines*. Make $Ab = \text{arch AB}$ (fig. 37.) 38.
and erect the ordinate $bp = \text{fine BP}$, then the flux-
ion of the area $= y\dot{z} = r\dot{v}$, and the fluent is $= rv$
 $r \times \text{AP}$, for the figure of *sines*.

And when Ab is a quadrant, the area $= rr$.

And when Ab is a semi-circle, the area $= 2rr$.

3. For the *figure of cosines*. Make $Ab = \text{arch}$ 39.
 B , (Fig. 37.) and at b , erect $bp = \text{CP}$ the cosine;

then the fluxion of the area is $= x\dot{z} = x \times \frac{r\dot{v}}{y} =$

$\frac{rxx}{y} = \frac{-rxx}{\sqrt{rr-xx}}$, and the fluent is $= r\sqrt{rr-xx}$

4. When Ad is a quadrant, the area $= rr$,
and there the curve ap intersects the axis at d , and
forms the same figure on the other side. This Fig.
the same as Fig. 38, beginning at the center, that
 $dbp = Abp$.

When Ab is a semi-circle the whole figure $= 2rr$,
in Art. 2.

4. For the *figure of tangents*. Make $Ab = \text{arch}$ 40.
 B (Fig. 37.) and at b erect the ordinate $bt = \text{the}$
tangent AT . Then the fluxion of the area $= t\dot{z}$.

But by Trigonometry, $x : y :: r : t = \frac{ry}{x}$; then $t\dot{z}$

$= \frac{ry\dot{z}}{x} = \frac{ry}{x} \times \frac{r\dot{v}}{y} = \frac{rr\dot{v}}{x} = \frac{rr\dot{v}}{r-v} = \frac{-rr\dot{x}}{x}$,

and the fluent $= rr \times \frac{2.30258}{-1} \times \log \frac{r-v}{r}$. But
when

Fig. when the area = 0, $v = 0$, and the fluent

40. rected is $2.30258 \times \log. \frac{r}{r-v}$ for the area.

Or thus; fluent of $-\frac{rr\dot{x}}{x} = -rr \times 2.30258$
 x ; but when the area is 0, $x = r$, therefore
 correct fluent = $2.30258 \times \log. \frac{r}{x}$.

Otherwise thus.

Since $x : y :: r : t$, therefore $xx : yy :: rr : tt$
 and $xx : xx + yy (rr) :: rr : rr + tt$. Whence
 $rr + tt = r^2$, in Fluxions, $2xx \times \dot{r} + \dot{t}t$, + xx
 $xx = 0$, and $\dot{t}t \times x = -\dot{x} \times rr + \dot{t}t$, therefore

$$\frac{\dot{t}t}{rr + tt} = \frac{-\dot{x}}{x}. \text{ But } \dot{t}t = \frac{-rr\dot{x}}{x}, \text{ therefore}$$

$\frac{rr\dot{t}t}{rr + tt}$ the fluxion of the area; and (Form 4th) the

fluent is $2.30258 rr \times \log. \frac{rr + tt}{rr} = \text{area Akt. 0}$

$$2.3025 \log. \frac{rr}{rr} \times rr = \text{that area.}$$

Since the tangent of 90 degrees is infinite,
 Ad be a quadrant, the ordinate dc will be an asymptote to the curve.

It is evident when Ad is a quadrant, the area Akt. is infinite.

41. 5. For the figure of secants. Make Ab = arc

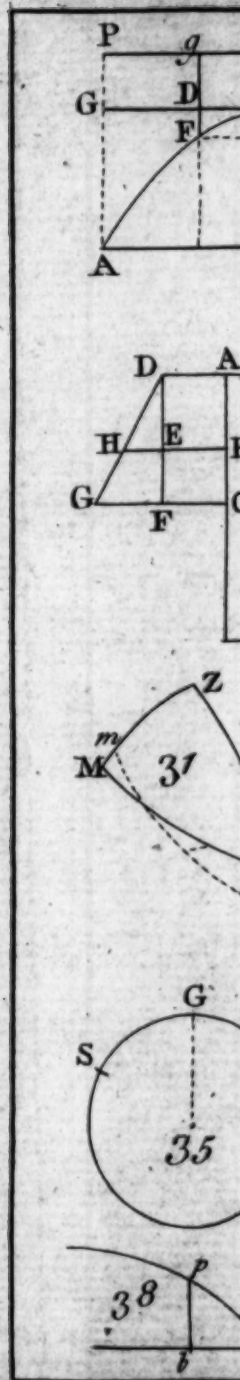
AB, and ordinate bp = secant CT. Then the fluxion of the area = $f\dot{x}$; and by Trigonometry

$x : r :: r : f$, and $f\dot{x} = rr$, in fluxions $f\dot{x} + x\dot{f} =$

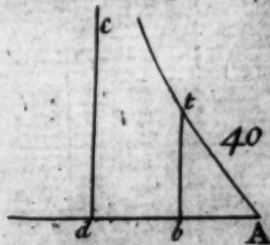
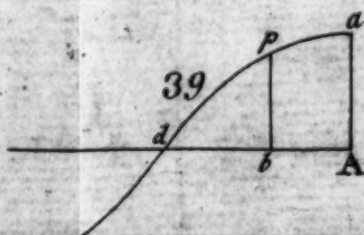
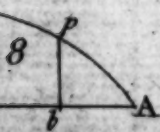
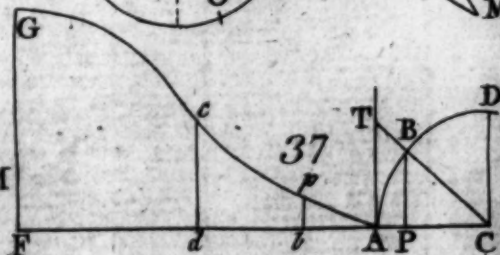
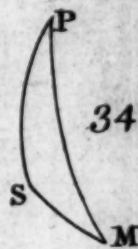
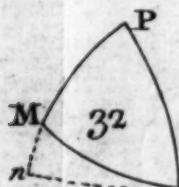
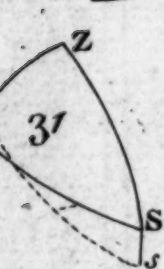
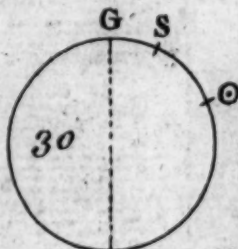
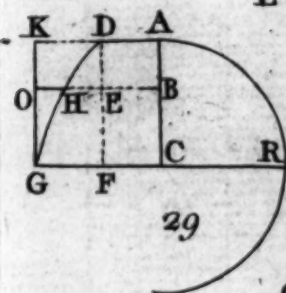
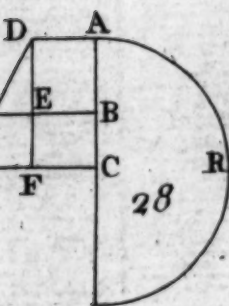
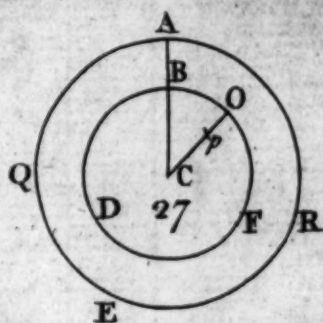
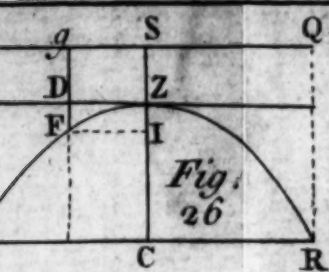
$$\text{and } \dot{x} = \frac{-x\dot{f}}{f}; \text{ and } \dot{z} = \frac{r\dot{v}}{y} = \frac{-r\dot{x}}{y} = \frac{-r}{y}$$



Miscellaneous



Miscellaneous



$\dot{f} = \frac{rx}{f}$, and $f\dot{z} = \frac{x}{y} \times r\dot{f}$. But $\frac{x}{y} = \frac{r}{t}$ Fig. 41,

$\frac{r}{\sqrt{ff-rr}}$; whence $f\dot{z} = \frac{rr\dot{f}}{\sqrt{ff-rr}}$. And (by

Art. 9th, Fluxions,) the fluent of $\frac{\dot{f}}{\sqrt{-rr+ff}} =$

2.30258 $\log. \sqrt{ff} + \sqrt{-rr+ff}$. But when the

area is = 0, $f = r$; therefore the fluent corrected

2.30258 $\times \log. \frac{f + \sqrt{ff-rr}}{r}$; and the fluent of

$\frac{rr\dot{f}}{\sqrt{ff-rr}} = 2.30258rr \times \log. \frac{f + \sqrt{ff-rr}}{r}$; for the

area $Abpa$.

If Ad be a quadrant, the ordinate dc will be an asymptote to the curve; and the area $Adca$ will be finite.

From hence the meridional parts in Mercator's chart may be calculated for any latitude AB or Ab . For the mer. parts for any latitude : is to the arch of latitude Ab : : as the sum of the secants : to the sum of as many radii : ; or as the area $Aapb$: to $Ab \times \text{radius } Aa$.

6. It is evident the area made by erecting the radii at all points of the arch AB is = zr or $Ab \times r$.

PROP. II.

To find the sum of the squares of all the sines, cosines, versed sines, &c. erected upon every point of the arch AB ; or to find the solidities of the bodies formed thereby.

1. For the versed sines. Denoting the quantities 37th as in the first Prop. we shall have $vv\dot{z}$ for the fluxion

Fig.

37. fluxion of the solid; but $\dot{z} = \frac{rv}{y} = \frac{rv}{\sqrt{2rv-vv}}$

$$\text{whence } vv\dot{z} = \frac{rvv\dot{v}}{\sqrt{2rv-vv}} = \frac{rv\frac{1}{2}\dot{v}}{\sqrt{2r-v}}. \text{ But (}$$

Form 10th.) fluent of $\frac{v-\frac{1}{2}v}{\sqrt{2r-v}} = \frac{AB}{r}$, and fluent

$$\text{of } \frac{v^{\frac{3}{2}}\dot{v}}{\sqrt{2r-v}} = \frac{4rr}{1} \times \frac{\frac{3}{2} \cdot \frac{1}{2}}{2 \cdot 1} \times \frac{AB}{r} + \frac{v^{\frac{3}{2}}\sqrt{2r-v}}{-2}$$

$$- \frac{\frac{3}{2} \cdot 2r}{-v} A = \frac{3}{2}r \times AB - \frac{v\sqrt{2rv-vv}}{2} + \frac{3r}{v} \times$$

$$\frac{-v}{2} \sqrt{2rv-vv}, \text{ and the fluent of } \frac{rv\frac{1}{2}\dot{v}}{\sqrt{2r-v}}$$

$$\frac{rv\frac{1}{2}\dot{v}}{\sqrt{2rv-vv}} = \frac{3}{2}rr \times AB - \frac{3rr+rv}{2} \sqrt{2rv-vv}$$

for the solid content of all the versed sines.

When AB is a quadrant, $v = r$, and the solidity

$$= 3r^3 \times \frac{3.1416}{4} - 2r^3.$$

And when AB is a semi-circle, the solidity =

$$\frac{3}{4}r^3 \times 3.1416.$$

38. 2. For the sines. The fluxion of the solid = $yy\dot{z}$,

$$\text{but by the nature of the circle } \dot{z} = \frac{ry}{x} = \frac{ry}{\sqrt{rr-yy}}$$

$$\text{whence } yy\dot{z} = \frac{ry^2\dot{y}}{\sqrt{rr-yy}}. \text{ But (Form 10th.) fluent}$$

$$\text{of } \frac{y\dot{y}}{\sqrt{rr-yy}} = \frac{z}{r}, \text{ and (Form 11th.) fluent of}$$

$$\frac{y^2\dot{y}}{\sqrt{rr-yy}} = \frac{rz}{2} - \frac{y\sqrt{rr-yy}}{2}, \text{ and fluent of}$$

$$\frac{ry^2\dot{y}}{\sqrt{rr-yy}} = \frac{rz-xy}{2} r = \text{area APB} \times r \text{ (Fig. 37.)}$$

for the solid of sines.

When

.XL. FIGURE OF SINES, &c. 237

When Ab is a quadrant, the solidity = quadrant $Fig.$
 $D \times r = .7854r^3.$ 38.

When Ab is a semi-circle, the solidity =
 $\frac{16}{3} r^3.$

For the cosines. The fluxion of the solid is 39.

= $xx \times \frac{ry}{x} = rxy = ry \sqrt{rr-yy}$; the fluent

$\frac{y}{\sqrt{rr-yy}} = \frac{\text{arch } AB}{r}$, by Form 10th. And

that of $y\sqrt{rr-yy} = \frac{2 \times \frac{1}{2} rr}{2} \phi + \frac{y\sqrt{rr-yy}}{2}$;

(Form 13.) fluent of $ry\sqrt{rr-yy} = \frac{r \times AB + xy}{2}$

$\frac{rz+xy}{2} r$, for the solid of cosines (Fig. 37.)

When Ad is a quadrant, the solidity = $\frac{rr \times AD}{2}$

quadrant $AD \times r.$

For the solid of tangents. Here the fluxion of 40.

solid is $tt\dot{z} = \frac{ttr\dot{v}}{y} = \frac{-ttr\dot{x}}{y}$. But by Art.

of the last Prop. — $\dot{x} = \frac{\dot{x}ti}{rr+tt}$, and $\frac{x}{y} = \frac{r}{t}$,

therefore $tt\dot{z} = \frac{-ttr\dot{x}}{y} = \frac{ttr}{y} \times \frac{\dot{x}ti}{rr+tt} = \frac{x}{y} \times$

$\frac{t}{-tt} = \frac{r}{t} \times \frac{rt\dot{t}}{rr+tt} = \frac{rt\dot{t}}{rr+tt}$. But (by Form

the fluent of $\frac{\dot{t}}{rr+tt} = \frac{AB}{rr}$; and (Form 11th),

fluent of $\frac{rt\dot{t}}{rr+tt} = \frac{-rr}{1} \phi + \frac{1.t}{1} = -AB +$

$t =$

Fig.

40. $t = t - z$, and the fluent of $\frac{rrt^2}{rr+tt} = \overline{t-z} \times$

$= \overline{AF - AB} \times rr$, for the solid of tangents (Fig. 37.)

When Ad is a quadrant, the solidity is infinite because the tangent is infinite.

5. For the secants. The fluxion of the solid

41. $\dot{ss}z = \frac{-\dot{ss}r\dot{z}}{y}$. But by the 5th Art. last Prop. —

$= \frac{\dot{z}f}{f}$, and $\frac{\dot{z}}{y} = \frac{r}{\sqrt{ss-rr}}$. Wherefore $\dot{ss}z =$

$\frac{-r\dot{ss}z}{y} = \frac{r\dot{ss}}{y} \times \frac{\dot{z}f}{f} = \frac{\dot{z}}{y} \times r\dot{ss} = \frac{rr\dot{ss}}{\sqrt{ss-rr}}$, and

the fluent is $rr\sqrt{ss-rr} = rrt$, for the solid secants.

It is plain when the arch AB or Ab , that is A is a quadrant, the solid is infinite.

6. For the sum of all the squares of the radii erected on the arch AB , we have rrz for the fluxion of the solid; and the fluent is rrz for the sum of all the radii square, or the solid formed thereby.

Cor. 1. It follows from the three last Articles, that the solid of secants is equal to these two solids, the solid of tangents, and the solid formed of all the radii square.

Cor. 2. Hence also the sum of the squares of all the sines in the quadrant = half the sum of the squares of as many radii. From Art. 2. and 6.

P R O P. III.

To find the sum of all the sines, cosines, versed sines &c. erected upon every point of the spherical surface, or to find the solids made thereby.

42. Here we suppose the circle ABD to revolve about the axis AC , and to describe a spherical surface.

and that upon the annulus described by BP, there Fig.
erected at every point of it, perpendiculars equal 42.

BP for the sines, or CP for the cosines, &c. and
for every annulus. But this is to be understood,
that every annulus is laid down in a plain, upon
an abscissa equal to the arch AB; then each annu-
lus will be the base upon which we are to erect the
perpendiculars. Then these form so many so-
lid bodies whose contents we must investigate.

1. *For the versed sines.* Supposing the quantities
denoted as in Prop. I. and let $c = 3.1416$, then
 yz is the area of the annulus at B; multiply this
by the versed sine v , and we have $2cvyz =$ fluxion
of the solid. But by nature of the circle $yz = rv$,
therefore the fluxion of the solid $= 2crvv$, and the
fluent $= crvv$, for the solid formed by all the versed
sines standing on the spherical surface described
by APB.

When the arch AB becomes the quadrant AD,
then the solid $= cr^3$: for then $v = r$.

And when the arch becomes a semi-circle, or the
surface, that of the whole sphere, the solidity $=$
 cr^3 . For then $c = 2r$.

2. *For the sines.* The general area of the annulus
AP is $2cyz$; which multiplied by the sine y , is
 $2cy^2z$; expunge yz , by taking its equal rv ; then
the fluxion of the solid is $2cryv$. But the fluent
of $yv =$ area BAP. Therefore the fluent of $2cryv$
 $= 2cr \times$ area ABP, for the solidity.

And for the hemisphere, the solidity $= 2cr \times$
quadrant ADC $= \frac{ccr^3}{2}$, for the solid of all the sines,

upon the hemisphere.

And consequently the sum of all the sines upon
the whole sphere $= ccr^3$.

3. *For*

Fig. 3. *For the cosines.* The area of the annulus BP is $2cy\dot{z}$, which multiplied by the cosine x is $2cyxz$; put $-rx = y\dot{z}$, and the fluxion of the solid is $-2crx\dot{x}$, and the fluent is $-crxx$. But at A the solid $= 0$, when $x = r$, therefore the fluent corrected is $cr^3 - crxx = cr \times rr - xx = cryy$, for the solidity. And hence the sum of all the cosines upon the hemisphere $= cr^3$.

4. *For the sum of all the tangents.* The area of the annulus BP $= 2cy\dot{z}$, which multiply by the tangent t , then $2cyl\dot{z} =$ fluxion of the solid, but $\dot{z} = \frac{ry}{x}$; therefore the fluxion of the solid $=$

$$\frac{2crty\dot{y}}{x} = \frac{2crty\dot{y}}{\sqrt{rr-yy}}. \quad \text{But by Trigonometry}$$

$$(\sqrt{rr-yy}) : y :: r : t = \frac{ry}{\sqrt{rr-yy}}, \quad \text{therefore the}$$

$$\text{fluxion of the solid} = \frac{2cry\dot{y}}{\sqrt{rr-yy}} \times \frac{ry}{\sqrt{rr-yy}} =$$

$$\frac{2crty^2\dot{y}}{rr-yy} = 2cr^4 \times \frac{\dot{y}}{rr-yy} - 2crty\dot{y}. \quad \text{But (Form 6,}$$

$$\text{the fluent of } \frac{\dot{y}}{rr-yy} = \frac{2.30258}{2r} \log. \frac{r+y}{r-y}. \quad \text{There}$$

$$\text{fore the fluent} = 2.30258 cr^3 \times \log. \frac{r+y}{r-y} - 2crty$$

the solid content of all the tangents.

And in the hemisphere, the sum of all the tangents is infinite.

5. *For the sum of the secants.* Here $2cy\dot{z} =$ area of the annulus, and by Trigonometry $x : r :: r : \dot{x}$ and $\dot{x} = rr$, in fluxions $\dot{x}\dot{x} + x\dot{f} = 0$, and $\dot{x} = \frac{-x\dot{f}}{f}$; but $\dot{z} = \frac{-r\dot{x}}{y}$; whence the fluxion of the

solid

$$\text{Solid} = 2cys\dot{x} = -2cfr\dot{x} = -2cfr \times \frac{-\dot{x}f}{f} = \text{Fig. 42.}$$

$$cr\dot{x}f = 2cr\dot{f} \times \frac{rr}{f} = \frac{2cr^3 \times \dot{f}}{f}, \text{ and the fluent} =$$

$cr^3 \times 2.3025 \log. f$, for the solid. But when the solid is 0, $f = r$; therefore the fluent corrected =

$$cr^3 \times 2.30258 \log. \frac{f}{r}, \text{ for the sum of all the se-$$

cants on the surface ABP.

And in the hemisphere, the sum of all the secants infinite.

6. *For the sum of all the radii.* The annulus = $yr\dot{x} = 2cr\dot{x}$. And the fluxion of the solid = $r^2\dot{x}$. And the fluent is $2crrx$, for the sum of the radii upon the surface described by BAP.

And the sum of all the radii upon the hemisphere, is $2cr^3$; and upon the whole sphere $4cr^3$.

Cor. 1. *The sum of all the versed sines on the hemisphere is equal to all the cosines on the hemisphere.*

Cor. 2. *The sum of the sines, cosines, and radii, on the hemisphere, are as $\frac{1}{2}c$, 1 and 2.*

For they are as $\frac{ccr^3}{2}$, cr^3 , and $2cr^3$.

R

A R T.

Fig.

A R T. XII.

FORTIFICATION.

FORTIFICATION is an art that teaches how to fortify any place, city, or town, by inclosing it with walls and other works to make it secure against the attack of an enemy; where a small number of men may defend themselves against an army. It is called *regular fortification* when it is built upon a regular polygon, as a pentagon, hexagon, heptagon, &c. and *irregular fortification*, when the figure is irregular, the parts not uniform, the angles and sides unequal.

DEFINITIONS.

DEF. I.

43. The *polygon* is the figure upon which the fortification is built; the *exterior polygon* is the figure of the outside, and the *interior polygon* is the figure of the inside. Thus ABOPQ is the exterior polygon and GKLMN is the interior polygon, GK the side of it.

DEF. II.

A *bastion* is a great work or bulwark built before an angle of the polygon, standing out towards the field, as FDAEH. It is built of stone or brick or sometimes of earth. There are several sorts of these, as a *full bastion* is one quite filled up with earth. A *hollow bastion* is that wherein an empty space is left, to hold magazines, provisions, &c. A *flat bastion* or *plat bastion*, is that made upon a right line and not at an angle. *Irregular bastion*

whose parts are out of due proportion. A *cut bastion* is one with two points. A *detached bastion*, 43. is separated from the body of the other works. A *demibastion* is but half a bastion, as ADFG. A *double bastion* is one built upon another; or one with another.

D E F. III.

The *demigorge* or *gorge line*, is a part of the side of the interior polygon, from the outside to the center of the bastion, as FG.

D E F. IV.

The *face* is the outside of the bastion facing the field, as DA.

D E F. V.

The *flank* is that part of the bastion joining to the face, as DF. It is called a *right flank* when perpendicular to the curtain. An *oblique flank* is one that makes oblique angles with the curtain. A *retired* or *covered flank* is one made hollow. The shoulder of the flank is the point D. A *second flank* is in a part of the curtain.

D E F. VI.

The *capital* is a part of the radius of the exterior polygon, contained between the angle of the interior polygon, and the extreme point of the bastion, as GA.

D E F. VII.

The *curtain* is that part of a side of the interior polygon, that lies between two bastions, as HI.

D E F. VIII.

The *line of defence* is the line drawn from the end of the flank to the extreme point of the bastion, AI.

R 2

D E F.

FORTIFICATION.

D E F. IX.

The *angle at the center*, is the angle that a side of the polygon subtends, as GCK.

D E F. X.

The *angle of the polygon* is the angle made by two sides of the polygon, as NGK. And the *angle of the triangle* is CGK half the angle of the polygon.

D E F. XI.

The *angle of the bastion*, is the angle made by the two faces of the bastion, as DAE. Some call it the flanked angle.

D E F. XII.

The *angle of the shoulder*, is that made by the face and the flank, as ADF.

D E F. XIII.

The *angle of the flank*, is that made by the flank and the curtain, as EHI.

D E F. XIV.

The *angle of tenail*, is that made by two lines of defence, as ATB, called also the *flanking angle*.

D E F. XV.

Angle diminished, is the angle which the flank subtends at the opposite end of the curtain, as EIH.

D E F. XVI.

A *line raissant* is a line running in direction of another line, as the face of the bastion.

D E F. XVII.

A *re entrant angle*, is that which has its point inwards; it is also called a *tenaile angle*. And a

an angle is one that looks, or points with its legs, Fig. 43. towards, or has the angular point outwards, contrary to the entrant angle.

Besides GCH is the *flank-forming angle*; and GI the *lengthened curtain*.

General Rules and Maxims in FORTIFICATION.

I.

Every part of a fortified place ought to be defended from another place, and therefore ought to be open to its view. For if any part of the fortification is not flanked or defended, the enemy could get there and be covered; and so the place would be more easily taken.

II.

The line of defence, or which is nearly the same as the side of the interior polygon, ought to be within musket shot; and therefore ought to be not more than 700 feet, or between 500 and 800 feet. For though both cannons and muskets are used, yet the length ought to be adapted to the reach of a musket; because a cannon requires a deal of time and labour to charge and discharge; and consumes a great deal of powder; often makes random shots; and is liable to be dismounted by the enemy's battery; so that muskets must be provided for.

III.

The line of defence must be a line raisant, when there is no second flank; and must terminate in the angle of the flank. When the face of the bastion continued cuts the curtain, the space between this point of intersection and the angle of the flank, is called the *second flank*. And the line of defence must never intersect the flank, for then

Fig. there would be a part of the flank, from which the face of the bastion could not be seen; the bastion ought to be defended from the whole flank.

IV.

The flanks, demi-gorges and second flanks, ought to be made as great as possible, provided the rest of the works are not injured thereby. For then they are fitter for defending a place, and can hold more soldiers, and more cannon,

V.

The angle of the bastion ought to be about 70 or 80 degrees, not so well when 90. Here regard is to be had to the number of bastions or sides of the polygon. For a less number of bastions require a more acute angle; and a greater number a more obtuse one. When the angle of the bastion is more acute, it is so much the sooner battered to pieces by the enemy. The bastion itself ought to be of a middling size, neither too great nor too little, great ones are too expensive, and small ones want room and convenience. The face ought never to be less than half the curtain.

VI.

It is better to have a second flank than otherwise. For by this means more room will be gained for fire. Also by this one may shoot further into the breach made in the bastion, and hinder the enemy from lodging there; and one may do more hurt to the enemy passing the ditch, as more of the ditch can be discovered.

VII.

The flank ought to be perpendicular to the raised line, or rather something greater than a right angle, otherwise it ought to be drawn to the centre.

of the place. Some draw it perpendicular to the face, but this makes the gorge too narrow. Others make it perpendicular to the curtain, but this is attended with the same inconvenience. And both these methods make the flank too oblique. But if the angle be much greater than a right angle, the flank is too much exposed to the enemy's cannon.

VIII.

The greater the angle between the side of the outward polygon and the face, the better is the face defended; for it is better seen from the flank; and besides, a second flank may be obtained by this means.

IX.

A part of the flank ought to be covered. As the flank is the chief part that defends the place, and fights for its safety; every thing that can, ought to be done for its strength and preservation, and to hinder it from being demolished. Therefore an *orillon* is very useful to cover the flank. An *orillon* is an additional part of the bastion, being built farther out at the shoulder, by which means the flank will be hollow, and is better secured from the enemy's shot.

X.

A fort ought to be equally strong throughout; and be so high as to command all places round about. And all outworks must be lower than the body of the place. If a fort be weaker in one place than another, the enemy will certainly attack in that place. If any place be out of sight of the fort, the enemy could cover his designs, and carry on his approaches unnoticed.

XI.

The works that are most remote from the center of the place, ought to be open to those that are

Fig. nearer. And therefore those that are nearest the center ought to be higher than those that are furthest off. For those that are furthest off, are most liable to be taken by the enemy; but lying open to the fort they may more easily be defended, and the enemy hindered from covering himself therein.

XII.

The more acute the angle at the center is, the stronger the place is; for then there will be more sides of the polygon.

XIII.

All these maxims must be so practised, as to consist and agree with one another, as far as possible. Yet it often happens when we adhere too strictly to one maxim, we deviate considerably from another. Thus, if we increase the second flank, either the opposite face of the bastion must be increased, the flank diminished; if the angle of the bastion be made very great, the defence it has from the flank or second flank is diminished, and it is more exposed to the enemy's cannon. If the gorge be made great, it causes the face of the bastion to be too great. There is always advantage attended with disadvantage, and our reason must judge how far we may stretch one maxim, and not encroach too much upon another; but endeavour to make them all agree as near as can be.

Likewise the expence of the works is a material article, and ought to be well considered; lest more money be laid out upon the defence of a place, than all the advantages of it amount to.

PROB. I.

Fig.

To describe a regular fortification about any place
even.

It very seldom happens that a place is regular of
self, yet when there is field room enough about it,
we may make it a regular one.

In the first place fix the side of the figure, as sup-
pose 700 feet, then divide the circumference of the
place by that number, and the quotient gives the
number of bastions. The quotient will seldom
come out a whole number, therefore make the side
something more or less than 700, that the quotient
may come out an integer. Suppose then there are
sides, or that the figure is a regular pentagon. 43.
The angle at the center GCK is $= 72^\circ$, and the
half is 36° . Then $S. 36 : \frac{1}{2} GK (350) :: \text{rad.} : CG$
the radius of the circumscribing circle 595. There-
fore with the radius 595 describe a circle from the
center C. And in this circle inscribe the pentagon
KLMN, each side being 700 feet. Produce CG,
that GA may be 194 feet, or thereabouts, for
the capital. From G set off GF and GH = 146
for the demi-gorge. Through F and H draw
FD, CHE, and set off FD, HE = 117 feet,
for the flank. Then draw AD and AE, for the
face. Or else set off the demi-gorges NS and KI
as before; then through D and E draw SDA and
SEA to intersect in A, the angle of the bastion.
Or the flanks EH, DF may be otherwise drawn, to
make a certain angle with the curtain HI and FS;
or with the rasant lines IA, SA. And thus if at
each of the other angles K, L, M, N, a bastion
be drawn; then the fortification will be finished,
when the counterscarp or outside of the ditch is
drawn, *abdef*, &c. being parallel to the faces of
the bastions, 20 fathom wide.

If

Fig. If instead of drawing AD, AE to S and I, the
43. be drawn to some point of the curtain, then you
will have a second flank.

Note, all these things may be laid down on the
ground as well as on paper, by the help of the
Propositions in Sect. I. of the Surveying. And to
save the labour of making a deal of calculation
I have here inserted (from Mr. Ozanam) a Table
containing all the lines and angles of a fortified
place, from the square to the dodecagon, suppos-
ing the side of the interior polygon 120 fathoms.
As for triangles, they are hardly to be fortified any
other way but with half bastions.

polygons	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
angle at the center	0° 0'	0° 0'	0° 0'	0° 0'	0° 0'	0° 0'	0° 0'	0° 0'	0° 0'
angle of the polygon	90° 0'	108° 0'	120° 0'	128° 34'	135° 0'	140° 0'	144° 0'	147° 16'	150° 0'
angle of the flank	120° 58'	112° 59'	108° 7'	105° 7'	102° 28'	100° 39'	99° 16'	98° 34'	97° 45'
angle of the should.	130° 40'	126° 18'	124° 56'	125° 21'	126° 14'	127° 55'	130° 0'	129° 21'	128° 34'
angle of bastion	70° 36'	81° 22'	86° 22'	88° 6'	87° 28'	85° 28'	82° 34'	85° 31'	87° 15'
demi-gorge	24° 0'	25° 0'	26° 0'	27° 0'	28° 0'	29° 0'	30° 0'	30° 0'	30° 0'
curtain	72° 0'	70° 0'	68° 0'	66° 0'	64° 0'	62° 0'	60° 0'	60° 0'	60° 0'
flank	16° 0'	20° 0'	24° 0'	28° 0'	32° 0'	36° 0'	40° 0'	40° 0'	40° 0'
line of defence	F f 117 : 3	f 117 : 5	f 119.	f 120 : 3	f 122 : 5	f 126	f 129	f 127	f 125
face	36 : 1	37 : 5	40 : 1	42 : 2	45 : 2	48 : 5	52 : 3	50 : 0	49
capital	28.	33 : 3	39 : 4	46 : 1	53 : 4	61 : 2	69 : 4	67 : 5	67
radius	84 : 5	102 : 1	120	138 : 2	156 : 5	175 : 2	194 : 1	217 : 5	233

The Table above will be ready for laying down
the fortification of any place that has no more than
12 sides. And the computation of it is as follows
for any of the figures, as suppose a pentagon.

ART. XII. FORTIFICATION.

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1. The side of the polygon GK being 120 fa- Fig.
horns, the demi-gorge FG is 25 for the pentagon, 43.
the flank FD = 20.

From the side of the polygon GK 120
subtract twice the demi-gorge GH 50

2. Remains for the curtain 70

3. If 360 be divided by 5 the number of sides,
the quotient is 72 the angle at the center GCK.

And 72 taken from 180, leaves 108, half of
which 54 is,

3. The angle at the base CGK, or half the angle
of the polygon.

4. To find the radius CG, by Trigonometry,

As cosine of CGK 54° 9.76921
to half GK 60 1.77815
rad. 10.

CG 102.1 2.00894

5. To find the angle of the flank DFS or CFG,
CG + GF = 127.1, and CG - GF = 77.1, and
180 - FGC (54) = 126 = sum of the angles F
and C, the half is 63.

Then sum sides 127.1 2.10401
difference 77.1 1.88705

tan. $\frac{F + C}{2}$ (63) 10.29283
12 17988

tan. $\frac{F - C}{2}$, 49 59 10.07587

then 63 + 49 59 = 112 59 = angle DFS of
the flank,

and 63 - 49 59 = 13 : 1 = angle GCF.

6. To find the angle of the shoulder ADF or
SDF, SF + FD = 90, and SF - FD = 50, and
180 - DFG (113) = 67, half is 33 30 = $\frac{D+S}{2}$.

Sum

Fig.	<i>Sum sides</i>	90	1.95424
43.	<i>diff. sides</i>	50	1.69897
	<i>tan.</i>	33 30	9.82078
			<u>11.51975</u>

$$\text{tan. } \frac{D-S}{2}, \quad 20 \quad 12 \quad 9.56551, \text{ and}$$

$$33 \quad 30 + 20 \quad 12 = 53 \quad 42 = \text{FDS.}$$

Then $180 - 53 \quad 42 = 126 \quad 18 = \text{angle ADF of the shoulder.}$

$$\text{Also } 33 \quad 30 - 20 \quad 12 = 13 \quad 18 = \text{angle DSF,}$$

7. To find the angle of the bastion EAD. To ADC (126 18) add DCA or FCG (13 1), the sum is 139 19, which taken from 180° , leaves 40 41 for DAC half the angle of the bastion; therefore the angle of the bastion is 81 22.

8. To find the line of defence SA, and capital AG. $SG = SF + FG = 95$.

Then S.SAG (40 41)	9.81416
GS (95)	1.97772
S.AGF or SGC (54 00)	9.90795
	<u>11.88567</u>
SA 117.9	2.07151
S.ASG or DSF (13 18)	9.36182
	<u>11.33954</u>

$$\text{GA } 33.5 \quad 1.52538$$

So the line of defence is $117^F \quad 5^f$, and the capital $33^F \quad 3^f$.

9. To find the face AD. $CA = CG + GA = 135.6$

Then S.ADC or SDF (53 42)	9.90629
AC (135.6)	2.13225
S.ACD (13 1)	9.35263
	<u>11.48488</u>
AD 37.9	1.57859

So the face is 37 fathoms, 5 feet.

And after the same manner, the requisites may be calculated for any figure or polygon.

To construct the figure from the Table.

Fig.

43.

The side of the polygon being 120 fathom, the radius for the pentagon is $102^F : 1^f$; with this radius, and center C describe a circle, in which inscribe the pentagon GKLMN. Through the points G, K, L, M, N, draw the lines CA, CB, CO, CP, CQ. And set off the capital 33 fathom, 3 feet, from these points = GA, KB, LO, MP, NQ. Then the demi-gorge being 25; make GF, KH, NS, KI, &c. each = 25. Through F, H, &c. draw the lines CD, CE, &c. And the flank being 20, make FD, HE, &c. = 20. Then from S, I, &c. draw lines through D, E, &c. to intersect in A, &c. making DAE the angle of the bastion, and so of the rest, till all the parts be finished.

Note, if you would have the side of the polygon more or less than 120 fathom, it is but taking a length proportionally bigger or less than a fathom, and dividing it into 6 parts. Then all the numbers may be taken out of the Table the same as before, and set off, after this new measure or scale, either upon paper, or on the ground.

S C H O L.

We have laid down in this Prob. a general way of drawing a fortification, as is commonly practiced. But different engineers have different ways and measures, for every article here mentioned. I shall therefore mention the several methods that different writers have laid down for that purpose.

1. In the method before laid down, the faces A, EA of the bastion are formed by drawing lines from the angle of the flank, SA and IA. But some choose rather to make the angle DAE a right angle, by which means they will generally have a second flank, if the polygon has many sides. Now if

Fig. if $DAE = 90^\circ$, its half is 45° , which taken from

43. SGC, half the angle of the polygon, leaves the diminished angle ASG, to which add the angle of the flank DFS, gives the angle of the shoulder ADF. The second flank is found by producing AD to cut the curtain.

2. Others make the gorge-line FH equal to the capital GA. Here the demi-gorges FG, GH being given, and the angle of the polygon FGH, the right line FH will be found by plain Trigonometry; then $CG + FH (AG) = CA$. Then the angle of the flank DFS or GFC will be known by the Table; and in the triangle CFG, all the angles being given, and the sides FG, GC; the side CF will be found. To this add the flank FD (taken as you please), gives CD. In the triangle ACD, the sides AC, DC, and angle ACD (from the Table) being given; the side DA and angle DAC will be found; and $2DAC = DAE$ the angle of the bastion.

3. Others construct the Problem after this manner. Let P be = the length of the perpendicular drawn from the center C to the middle of any side of the polygon, n = number of sides of the polygon. Then make the demi-gorge GH or GF = $\frac{2}{n+1} P$, and the capital $AG = \frac{3}{n+1} P$. Then draw the raifant lines SA, IA; and through F and H, draw from the center C, the flanks FD, HE. This is a good construction. If you have a means to compute the quantities by Trigonometry, you will have this proportion,

radius,

tan. half the angle of the polygon KGC,

so half the side GK,

to the perpendicular P.

Then we shall have FG or GH, and AG. Then $GK - GH = HK$ or GI . Then in the triangle AGI, there is given AG, GI, and the angle AGI

find the angle GIA, and the angle GAI, or Fig. 43. If the angle of the bastion. The radius. CG 43. found by this proportion.

As the sine of half the angle of the polygon CGK, to the perpendicular P, so is radius,

to CG.

When AC is known. And in the triangle CGH, there is given CG, and GH, and the angle CGH; find the angle CHG or EHI, the angle of the flank, and the side CH, and the angle GCH or ACE, then the angle of the shoulder AEC is known. And in the triangle CAE, the side CA, and all the angles are given; to find the side AE, the face. Lastly, in the triangle IEH, all the angles are given, and the side HI; to find the flank HE.

It may be observed, that this method makes no second flanks.

4. Another way of fortifying is this. Let AB be the outward side of the polygon. Make the angles CAI, CBH, each 45 degrees. Bisect the angle CBH by the line BI, to cut AI in I; and draw IH parallel to AB, to cut BH in H; then IH is the curtain. Then draw the flank HE perpendicular to AI, and so of the rest; so the shoulder will be a right angle. This is to be performed from the square to the octagon. But in polygons of more sides, the flank HE must be perpendicular to the curtain IH. And all the requisites may be computed trigonometrically like the rest. But enough has been said about such computations.

5. This is another method of fortifying. Make the exterior side $BO = 180$ fathoms, to the middle of BO draw the perpendicular CR. Make $VR = 24$ fathoms (but in the square only 24); draw through V, the lines of defence OX, BY. Make the faces B_1, O_2 , each $= 55$ fathoms. Then draw

Fig. draw the flanks 1X, 2Y, perpendicular to the tangent lines XO, YB. And through the points of intersection X, Y, draw the curtain XY.

6. Another way of fortifying. Make the inward side of the polygon GK = 120 fathom; make GH and IK = a fifth part of GK, for the demi-gorges and FD, HE = a fourth part of GK, for the flanks. And make the angle of the flank SFD or IHE = 100 degrees. Through D and E draw the tangent lines SDA and IEA intersecting in A, the angle of the bastion, and the faces are DA and EA.

7. Or thus, if the inward side GK be 140 fathoms, the demi-gorge GH, IK must be 28; the curtain HI = 84, and the flank EH = 35; then make the angle of the flank EHI = 100 degrees. Then draw through E, the line of defence IEA; likewise draw the line of defence SDA, to intersect IA in A, the angle of the bastion, whose faces are AD, AE.

8. Otherwise thus; let n = number of sides of the polygon; then make the diminished angle $ASF = 45 - \frac{120}{n}$ degrees = angle ASF. Then the

lines IA and SA being drawn, the intersection gives the angle of the bastion at A. Or rather make the angle ABH, and $BAI = 45 - \frac{120}{n}$. Make BH

AI each = $\frac{1}{10}$ of AB. Draw HI for the curtain. Let AE = ET = AD, then AE, AD, will be the faces of the bastion. And draw EH for the flank, and so of the rest.

9. Or thus; upon the outward side BO let fall the perpendicular CR, which bisects BO; take $RV = \frac{1}{3}BO$ for the square, $\frac{1}{4}BO$ for the pentagon or $\frac{1}{5}BO$ for the hexagon, heptagon, &c. through R draw the lines of defence OVX, BVY. Make OZ, and BZ, each = $\frac{1}{3}BO$. And make 1Y, 2

12, a line drawn from 1 to 2. Then draw Fig. 43. for the curtain; and 1X, 2Y, for the flanks. 43. This is a very good method.

40. Or thus; make the inner side GK 130 fathom; make the demi-gorges GF, GH, each = 25, and likewise the flanks DF, EH = 25; and draw the flanks perpendicular to the curtain HI. And make the second flank 3 fathom. From each point draw a line through E or D, will intersect the radius CG produced; in A the point of bastion. But the square or pentagon are better without second flanks.

Or one may make the flanks and demi-gorges a fourth part of GK. And make the angle of the bastion DAE a right angle, especially in the hexagon, pentagon, &c. whence the capital will be equal to the gorge line.

There are several other methods of fortifying; and in all these methods each of them has some advantages, which the others have not; but which, to choose, must be left to the judgment of the engineer, who must take that method which best suits the particular purpose.

P R O B. II.

To construct an irregular fortification about a place.

Many places cannot be inclosed by a regular figure or a regular fortification; in which case all that can be done is to reduce it as near to a regular polygon as is possible; and then we must apply the rules of a regular fortification, as far as the nature of the place will admit of. The principal design in regular and irregular fortification is to make all the parts equally strong. For if some be weaker than others, they will certainly be attacked at the weakest places. Longer sides of a polygon

S

are

Fig. are stronger than shorter sides, provided they
 44. within musket shot. Also greater angles of a polygon are stronger than less or more acute angles. But to come to particulars.

1. Let GKLMN be an irregular polygon; find the center C of a circle whose circumference pass through three of the most distant points K, M, N. Draw lines from the center C to all the angles of the polygon, CK, CG, CN, &c. and measure the angles at the center KCG, GCN, &c. and compare them with the angles at the center in the general Table, to find which they come nearest. Then fortify that side by the rules of that polygon; that is, divide that side into 120 equal parts, which will serve for a scale; then set off the gorge and flank, by the numbers in the Table belonging to that polygon. As if the angle GCK be 46 degrees, 360 divided by 46 , give almost 8 for the quarters; therefore GK must be fortified as an octagon. And then the demi gorge will be 28, the flank 32, which must be drawn from the center C. After the same manner must the demi-gorges and flanks be made for the rest of the sides, and the bastions drawn.

2. Or thus; when the polygon is very irregular take two adjoining sides GK, GN; and find the center of a circle O, passing through the three points K, G, N. Then measure the angles KOG, GON. Then the flanks and demi-gorges are to be taken from the Table as before, according to the polygons they agree to, and the half bastions on each side compleated. When the polygon has an odd side; bisect the two angles at the ends of it by two right lines, whose intersection will give the center required; then make two half bastions before: a good method.

3. Otherwise thus; bisect all the angles of the irregular polygon by right lines. The two lines drawn from the two ends of any one side, will

their intersection, give the center of a supposed regular polygon, one side of which polygon is the included side. And that side is to be fortified according to the angle at the center. Thus if the angles K , be bisected by the lines GC , KC ; then C will be the center of the polygon, and GK a side of it. And the angle GCK shews by the Table what polygon it agrees to. And two semi-bastions are to be constructed at G and K .

4. Let GK be a side of the polygon. Bisect the angle K by the line KC , and bisect the side GK by the perpendicular mO , to intersect KC in n , and C be the center of a regular polygon whose side is GK ; and angle at the center, $2Knm$. According to which the demi-gorge and flank must be set off from the Table, and the half bastion made at K . Again, divide the angle G into two equal parts, by the line GO , to intersect mO in O ; which must be taken for the center of a regular polygon, whose side is GK , and angle at the center $2GOm$; according to which the demi-gorge and flank must be set off at G , from the Table, and the half bastion A made at G . And this work must be performed for all angles and sides in the polygon.

5. Otherwise. Let AB , AQ , be two sides of an exterior polygon, bisect them in s and w ; and draw the two lines sO and wP perpendicular to these sides AB , AQ , to intersect in p . Then p will be the center of a regular polygon, whose angle at the center is spw . Then for a hexagon or more sides, take $st = \frac{1}{2}AB$, and $wq = \frac{1}{2}AQ$, but take $\frac{1}{3}$ for a square, $\frac{1}{4}$ for a pentagon. Then the demi-gorges and flanks are to be set off from the Table; and the lines At and Aq drawn to form the bastion at A . And so of the rest.

6. To fortify a side that is too long, as MP . This side being too long to be defended by muskets, for reason that the bastions at M and P , are too far

Fig. distant to defend one another; there must be erect

44. ed in the middle of MP, a *flat bastion Z*, in which the gorge must be about 60 fathoms, and the flanks about 30. And the flanks must be perpendicular to the side MP. And the capital must be equal to the gorge, and the angle of the bastion a right angle. And thus two bastions may be placed on a side that is very long.

7. When a side is too long, but not long enough to require a flat bastion, as PL; it must be fortified, with a *curtain retired inwards* as F. In this case, the demi-gorges of the bastions at P and L are $\frac{1}{3}$ PL, and the flanks the same, which are perpendicular to the curtain, the retired curtain $F = \frac{2}{3}$ PL, and the flanks of it, $\frac{1}{3}$ LP. And the flanks must also be perpendicular to PL.

8. In all sorts of fortification, regular or irregular, for the greater security of the flanks, it is very common to make them hollow as in the bastion at R. This is called a *retired or covered flank* for it is covered by the butment at the shoulder called the *orillon*; and this is called a *round orillon* when it is round, and a *square orillon* when it is square. This flank is sometimes an arch of a circle described from the angle of the opposite bastion Q; and sometimes a right line, and then its ends are equally distant from Q. This retired flank is half the length of the whole flank, by some, and $\frac{2}{3}$ of it, by others, or more. And half of it is allowed for the depth; but some allow but half as much for the depth.

9. When an angle is too acute, it is easily battered down by the enemy's cannon; therefore it must be defended by some additional works. If the adjoining side be long, demi-bastions must be erected on these sides; the greater flank towards the angle, and near double the lesser flank; and both of them perpendicular to the side it stands

but if the angle be very acute, it may be fortified Fig. 44. by cutting off the angle, and making it into a *cut* 44. *bastion*, called a *bastion with a tenaille*; which will be two demi-bastions, where the longest flanks are at the angle; and these flanks must be perpendicular to the sides, as you see at M.

10. To fortify an inward angle (of which there is none in this figure); you may construct a *curtain* *tired inwards*, as is done at F, in the line PL. or you may make a platform before it, as V in the line KL. There are other ways of defending an angle; instead of making the platform a parallelogram, it may be made a triangle, &c.

11. Along a river side, or by the sea; it is usual to have an *indented line* made, which runs in and out; and these lines are called *saw-works*, as being angular like the teeth of a saw. The lines perpendicular to the side are the flanks, being about 25 fathom, and furnished with cannon. Many other methods may be used in fortifying irregular sides and angles, which the experience and judgment of the artist will easily find out.

As to the different situation of places, some are advantageous, and some the contrary, and want more help. In a town already walled about, the several parts of the wall will serve for curtains; and bastions are to be made at the angles; and likewise flat bastions, when the angles are too far under. If the wall be bad, and no ditch; it will be necessary to circumscribe the town with a new line, and then fortify it according to art.

When an eminence is very near the place, it must be taken in with the fortification, or else build a castle on it, to hinder the enemy from taking possession of it. But if it be in the enemy's possession; bastions must be made solid, and full, and the capets raised higher than ordinary; so that they may command such an eminence, if possible.

Fig. 44. When a town is situated upon a rock or a mountain, it is hardly accessible without great difficulty and therefore requires less fortifying; and it may be sufficient to fortify the avenues that lead to it which are generally very few.

If a town be situated near a river, that side near the river must be defended with a rampart, and if it be very long, must have one or more flat bastions placed on it; and sometimes out-works must be made the better to guard the river side. Or if the place be situated near the sea, it must be fortified by the same rules; and to hinder great ships from entering the harbour, bastions must be placed on the opposite sides.

Places that are situated in an island need not be regularly fortified; for the enemy can erect no batteries upon the water. Here the water is a natural fortification, and therefore requires less artificial work. And instead of bastions, redoubts that defend one another may be sufficient.

Places of a high situation, have a good air, see all the works of the enemy, and command all places round about, and therefore can do more execution. No works of the enemy can be raised but what may be easily demolished. The defenders can see without being seen. Mountains being naturally strong, are fortified more easily, and at less expence. But on the contrary, it is more difficult to shoot downwards with cannon; and there is often a want of water in such places. And these places often want room as well as regularity, which makes them very troublesome to fortify.

Places situated in a plain have plenty of field room, where one may fortify to the best advantage. It is more easy to get necessaries in such a place. But then the defenders have no more advantage from the situation of the place than the enemy has.

Places situated in vallies have all the disadvantages possible; they are encompassed with hills that command the place, whence the enemy will destroy all their works, dismount their cannon, and demolish every thing. The fortifying such a place to any purpose is an immense expence; and therefore it is better to abandon it.

Places situated in marshy grounds are well situated, for they can be attacked but in few places, which may be soon fortified. The enemy cannot get earth near to make batteries, nor can they make any mines. Yet such places having only soft earth, their bastions often sink or fall down. And sometimes the place may be laid under water; or the water that surrounds the place, may be let off and drained.

Places near a river or the sea, have this advantage, that near the water they are easily fortified. And the town may easily receive necessaries by shipping. Great ships cannot stay long to annoy the place for fear of storms; and it requires more force to besiege and take it. Yet the enemy has the same advantage of the sea as the besieged, and may prevent supplies coming to the town.

In islands situated at a good distance from the shore, the people have nothing to fear from the land. And they can easily hinder the descent of the enemy. Generally these places may be fortified with little expence. Yet they may be surprized by the enemies fleet, who may do them a great deal of mischief. But the town has an equal advantage in receiving succours by shipping.

The advantages of a good situation, are good air and good water. Bad air breeds the plague, and bad water the fever and scurvy. The place ought to be so high as to command without being commanded. It is necessary the earth be good to work with; that its situation may make it difficult

Fig. to besiege; and that it may be succoured in spite of the enemy. Near the sea a good harbour for shipping is one of the greatest advantages; and likewise places situated upon some large navigable river, not far from the sea, are proper for carrying on trade and commerce.

P R O B. III.

To describe the profile or section of a fortification its rampart, parapet, ditch, &c.

The RAMPART.

46. The rampart is a great bank of earth made round a town, to defend and secure it from the enemy, as $ABCDb$; the earth it is made of, is taken out of the ditch $EFGH$. The height of the rampart Ba is generally about 3 fathom, and its breadth at bottom Ab , about 15 fathom. The sides Ba and Db are sloped, because earth cannot stand if be built upright; the slope within at A is such that the base Aa is about equal to the height Ba and the slope without at D is but half as much, such that $cb = \frac{1}{2}Dc$. But these slopes depend much upon the consistence of the earth they are built with; for loose earth requires a greater base, or cannot stand. And the slope on the inside may be as great as you please, that the people may easily go up, or convey the guns to the top (called *terre plain*), for the defence of the place. The rampart is built upon some polygon, and at the corners or angles are placed the bastions also made of earth, as described before. Sometimes these bastions are solid earth and sometimes hollow, to serve for magazines to keep ammunition and provisions.

The PARAPET.

Upon the out edge of the rampart, or the side facing the field, there is raised a bank of earth called

called the *parapet*, or *breast-work*, CDc. This is Fig. 46.
 about 20 feet thick, and 6 feet high towards the
 face or on the inside, and 4 or 5 feet high towards
 the field. Which difference of height at C and
 D, makes a slope for the musketeers to fire down
 into the ditch; *d* is a *foot-bank* to step upon; on the
 inside is left a space *bE* called the *berm*, 3 or 4 feet
 wide, to receive the earth that falls from the para-
 pet, and to hinder it from falling into the moat.
 The use of the parapet is to cover the besieged,
 and to defend the men from the enemy's shot.
 All along the parapet at certain distances, there are
 openings called the *embrasures* for the cannon to fire
 through; they are generally about 12 feet distant.

The DITCH.

The *ditch* or *moat* is a hollow channel made be-
 yond the rampart, as EFGH, and goes round
 about the place. The level of the ground is EH;
 this ditch is made at the same time the rampart is
 made, for the rampart is built of the earth that
 comes out of it. The edges of the ditch EF and
 HG are made sloping; and the slope EF next the
 place is called the *scarp*; and that next the field
 HG is the *counterscarp*. The counterscarp is made
 round or circular over against the point of a bas-
 tion, the rest of it is parallel to the respective sides
 of the bastions. But the counterscarp sometimes
 means the covert-way and glacis. The breadth and
 depth of the ditch cannot well be determined, as
 it depends on several circumstances, as particularly
 upon the nature of the soil, whether it is dry or
 watery. But it must be wider than the longest
 trees can reach, and ought never to be less than
 20 feet.

The edge of the ditch towards the field, over
 against the middle of the curtain, where it is
 broader than over against a bastion, is terminated
 by

Fig. by a re-entrant angle, called the *angle of the counter*
 46. *ter/escarp*. So that each part may be seen and de-
 fended from the opposite bastion. Ditches may be
 either dry or filled with water; in dry ditches
 there is made a lesser ditch *f*, about the middle
 called a *cunette*, which must be sunk to the water.
 And in ditches full of water, stakes or banks of
 sand must be placed about the middle, to stop the
 enemy's boats.

The COVERT-WAY.

The *covert-way* is a way left upon level ground
 beyond the counterscarp, whose breadth is about
 fathoms, it goes all round by the edge of the moat
 it is denoted by HI; it is about 18 feet wide. At
 the end I is a breast-work or parapet 6 feet high
 as KI; and a foot-step at I. Upon the covert-way
 at a little distance from the parapet, *pallisades* are
 placed, which are great stakes set fast in the ground
 with sharp points at the top, and 4 or 5 feet high
 and so close as one may just put a musket thro'.

The GLACIS.

This is the part beyond the covert-way, as KL
 it is made sloping from the top of the parapet K
 to the level ground at L, being about 20 fathoms.
 The slope KL ought to be such, that the line LK
 continued will pass through C.

Besides the parts mentioned, there are some
 others that are sometimes made use of, as the *fal-
 bray*, which is a parapet made between the rampart
 and the ditch, about 6 feet high, and a way left
 between it and the rampart 3 fathoms wide, called
 the *chemin des rondes*. From this one may see what
 is done in the ditch.

PROB. IV.

Fig.

To describe the out-works of a fortification, and their uses.

All the works made beyond the ditch are called *out-works*, as *ravelins*, *half-moons*, *horn-works*, *horn-works*, &c. These are designed to cover the body of the place, and are made one before another; these that are nearer must be made higher, as to command those that are further off. The *out-works* are the most important places in a fortification, and are the main strength of it. For however strong a rampart may be, if it is not defended by out-works, it cannot stand out long, being continually battered by the enemy. But out-works stop the enemy, retard his progress, and gain time for the besieged. I shall here lay down the plan of each of these; for their heights differ, according to the situation of the place.

A RAVELIN.

A *ravelin* is a work placed upon the angle of the 45. counterescarp, over against the middle of the curtain, as CDE, the angle is towards the field; it is like the fore-part of a bastion with the flanks cut off. The two faces CD, DE, make an angle of 60 degrees or more. A, B, are two bastions, EFG the ditch. Its use is to cover the flanks of the bastions A, B; and also a bridge or a gate. There is a ditch on the outside of it, half the breadth of the great ditch. The faces CD, DE, point to the centers of the bastions A, B. It has a rampart and a parapet, to be cannon proof; and all these are parallel to the faces. They are generally made of earth, but sometimes for greater strength and last, they are walled, especially against gates.

Fig. gates and bridges, which ought to be well defended. Flanks are sometimes added to a ravelin, and then they must be perpendicular to the curtain but these flanks are of little advantage.

A HALF-MOON.

47. This is an out-work consisting of two faces which make an angle whose point is from the field. Its gorges bend in like a bow or crescent, and always placed before the bastion. ABCDE is half-moon, ED is the counterscarp, CD and AD are each = half the demi-gorge of the bastion and in the direction of the faces HI, KI; and AD, CB, must be parallel to the counterscarp at E and D, or to the faces of the bastion HI, KI. On the outside there is to be a ditch of the same bigness as in the ravelin. The use of this is to cover and defend the point of the bastion from the enemy's cannon.

A HORN-WORK.

48. *Horns* are placed before the ravelin, towards the field, to cover it and the curtain. These are made several ways, some have long sides parallel to one another and perpendicular to the curtain; these sides are called *wings*. Some grow narrower, and some wider towards the place. But the best way is to make them with two demi-bastions. From the shoulders of the bastions B, D, draw BA, DC perpendicular to the curtain LM, and equal to 120 fathom. Draw IK for the breadth, which divide into 3 equal parts at F, H. Making AI = IF, and KC = KH; raise the perpendiculars FE, HG, each = $\frac{1}{2}$ FH, and draw AE, GC, and AEFHGC is the horn-work, which is to cover the ravelin N. There is a ditch on the out-side, as in the other works. The horn-work must have a rampart and parapet, made of the earth that comes out of the ditch. It will be proper to place a ravelin

before the curtain of the horn, to cover its Fig.
flanks and curtain, constructed the same way as an- 48.
other ravelin before described.

A CROWN-WORK.

A *crown* is a kind of horn-work, made up of 49.
two demi-bastions, and a whole bastion in the mid-
dle. To construct a crown; from the shoulders
of the bastion, draw two lines AI, BC, perpendi-
cular to the curtain, and not so long as musket-
shot: perpendicular to them draw the line AB, so
that the angle AOI, and BOC may be 30 degrees;
AIO, BCO, 60 degrees; from O (the middle
of AB) describe a circle with radius OI, and make
the angles IOD, DOC, 60 degrees, or else ID,
C=IO. And let I, D, C, be the centers of
three bastions; at D make a whole bastion, and at
I, C, make two semi-bastions, such that the demi-
bastion and flank in each be a fourth part of ID, or
IC. It must have a ditch and rampart, like other
works. The use of these is to cover some large
spot of ground, to secure some rising ground from
the enemy. The sides AI, BC, are not drawn to
their full length, to save room. These are the
strongest sort of works that are made, and often
placed before a ravelin or other works. When the
sides AI, BC, are without musket-shot, shoulders
(or flanks) should be made in the middle of the
sides, which will serve to defend the crown-work.
There are other ways of making these crown-
works, as when the out-lines AI, BC, are not pa-
rallel, as when the work is broader at the crown
than towards the town, it is called a *swallow's-tail*.
There are also several other sorts of out-
works, as

Tenails; ABCDE is a *single tenail*, making a re- 50.
entrant angle at C. ABCDEFG is a *double tenail*, 51.
making two re-entrant angles at C, E. In both
these

Fig. these the sides AB, DE, or AB, FG, are parallel.

51. A *lunette*, is a work placed on both sides of a ravelin to strengthen it. It is almost in form of a ravelin.

Bonnet, a small work, consisting of two faces making an angle saliant, like a small ravelin.

Priests-cap, this is a swallow's-tail with two re-entrant angles; it consists of 3 saliant angles, and 2 re-entrant angles, it is used to take in springs and high ground.

Counter-guard, a work something like a ravelin and placed before a ravelin, or before a bastion and has its sides parallel to that ravelin or bastion.

Coronment, some small additional work to a crown.

Traverses, trenches with parapets, made slanting towards the enemy, and are designed to hinder him from passing through some narrow place or passage. And they serve to cover the pioneers. Traverses may be made many different ways, and generally run in several directions, making many angles.

Redoubts, these are little square works, with a ditch and parapet, but not so strong as to resist cannon. They are used to secure the lines of approach, and often made on the trenches to secure the workmen, or pioneers.

Lines of circumvallation, lines or trenches with a parapet, thrown up by the besiegers, round the place besieged, and at a cannon-shot distance. This is to cover and defend the besiegers.

Lines of contravallation, are trenches with a parapet, made nearer the town than the lines of circumvallation. Between these lines of circumvallation and contravallation, the soldiers lie that besiege the place.

Trenches or approaches, are works carried on by the besiegers, being cut into the ground, and parapets made of the earth. By these the besiegers

gain

rain ground, and draw nearer and nearer the place. Fig. And trenches made against the besiegers, to stop their progress, are called *counter-trenches*.

Fort, a small fortification, made at a small distance from a town; made to secure some spot of ground, or some pass; or to defend the trenches. They are of several forms, as triangular, square, pentagonal, &c. ABCD is a square fort with half bastions; and ABCDEF is a star fort with 6 points, made of saliant and re-entrant angles. It is a work entrenched on all sides.

A *citadel* or *castle*, is a small fort, erected in a town, containing 4, 5, or 6 bastions. It is placed on high ground to command the town, or hinder the approach of an enemy.

Caponiere, a lodgment sunk 4 or 5 feet into the ground, and raised 2 feet above ground. Here 15 or 20 musketeers are lodged, who fire through loop-holes made in the sides. These are generally made on the glacis or in a dry moat.

Cavaliers, a heap of earth raised high, at the top of which is a platform or battery, with a parapet to cover the cannon planted on it.

Battery or *platform*, a place to plant guns on, it is laid with planks, that the wheels of the carriages may not sink. They slope or descend a little towards the parapet, that the guns may not recoil much, and be more easily drawn back. The guns are 12 feet distant. A *sunk battery*, is one laid deep in the earth. *Cross batteries*, those that play athwart. *Murthering battery*, is one that beats upon the back of any place, &c.

Sappe, a deep trench sunk far into the ground, and descending by steps, and covered over head. This is used when there is a hot fire from the place, to avoid which, the besiegers work by sap, that is, under ground till they come to the covert-way and the ditch.

Gallery,

Fig. *Gallery*, a passage made from the sap, across the ditch, and covered over head with trees and earth. This leads to the place where the enemy intends to have a mine.

Mine, a hole dug under ground, being a long passage, with many turnings and windings in it; at the end is the *chamber* of the mine, made under some place designed to be blown up. In this is lodged a quantity of gun-powder for that purpose. The chamber is a hollow cube 6 feet every way. If the ground be wet, the powder is in barrels; if dry, in sacks. They are fired at once by several *saucidges* or pipes of tarred cloth, which reach from the mine to the place where the engineer stands springing the mine. The besieged also make such a hole and gallery under ground, to find the enemy's mine, and give it air; and this is called a *counter-mine*.

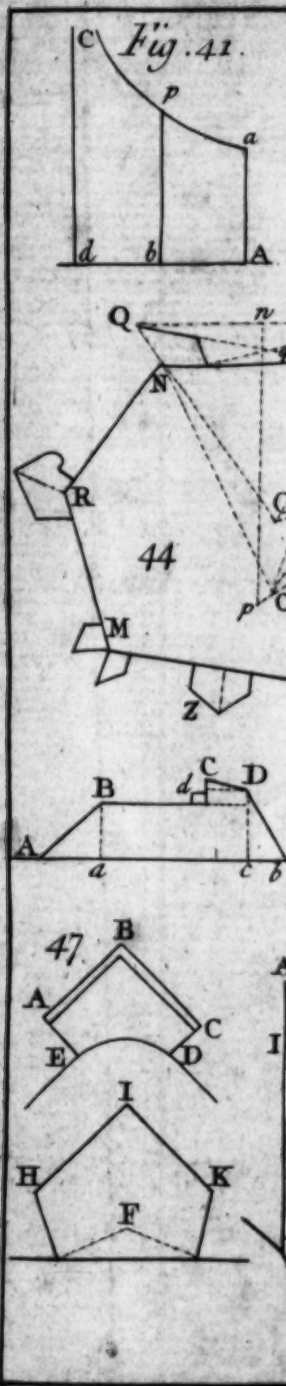
Enfiled, or *enfiladed*, the situation of a post that can see and scour all the length of a straight line or parapet, which by that means is rendered almost defenceless.

Thus I have described the principal out-works and other additional works of a fortification. These out-works are made one before another all manner of ways, to fill up the field. And their greatest use is to advance forward in the campaign, to hinder the sudden approaches of the enemy, and to spend his time before he comes to the main works.

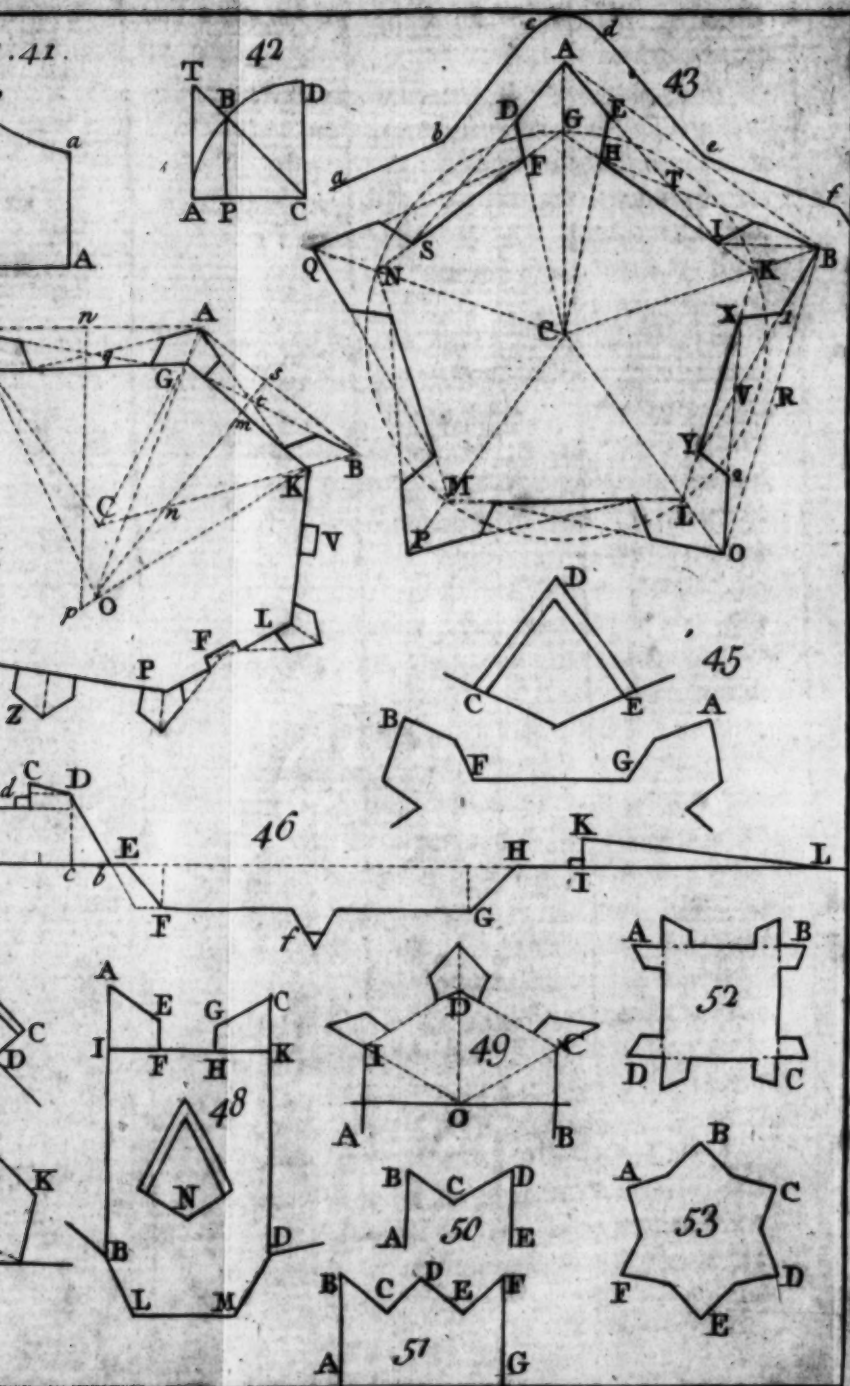
All ditches that are made, ought to communicate with one another, when filled with water; and then one will supply another.

SCHOOL.

If it be required to find the solidity of any of these works, and to cast up the expence required to do them. The content may be found by the rules of Mensuration, and it is no matter whether



Miscellanies.



rampart or ditch be measured, for one is nearly equal to the other, as the matter of the rampart is filled the ditch. But some engineers say, that earth dug out of a moat, being beat or rammed down, goes into a less room by $\frac{1}{11}$ part. Now suppose the ditch to be EFGH (Fig. 46.) the area of the section is easily found in square yards, and this multiplied by the length of the curtain or rampart, gives the solidity of the rampart for that length. If the breadth of the ditch be not the same all the way; take the greatest section and the least section, half the sum gives the mean section, for the length of the ditch. Now the section multiplied by the length, gives the solid yards in that ditch. And the like for every part, and for every ditch.

As to the expence, workmen will dig earth for one or three pence a yard; but if it is to be piled off, the price will be greater. Likewise time may be computed. For if a man can dig 5 yards in a day, 1000 men will dig 4 or 5 thousand yards. And the rule of proportion will shew how many days 1000 men will be in digging a number of yards.

Lastly, if you would represent any work in perspective, you must follow the rules laid down in p. I. *Perspective*, to which I refer the reader.

P R O B. V.

To describe the manner of besieging, and defending a town.

When proper fortifications have been made about a town, and a siege be feared; provision must be made for a sufficient quantity of victuals, according to the number of people, and the time of continuance. Also the town must provide plenty of

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arms

Fig. arms and ammunition, as cannon, muskets, powder and ball, &c. Also all sorts of implements of work with, as spades, shovels, pick-axes, carriage &c. and a sufficient number of soldiers to defend the place, as about 300 men for every bastion, that 2000 may be reckoned a good garrison, if the place is not very large.

The town being thus supplied with all necessities, and the flanks well lined with cannon. All the soldiers must be called to their particular posts, the inhabitants ordered to keep in their houses, observations must be made on all sides to discover the motions of the enemy.

When an army proposes to besiege a town, they are obliged to attack the out-works first. In order to this, they must pitch upon a proper place, without musket-shot, to begin the line of *contravallation* or *breaking ground*. This line is to be carried about the town, if the garrison be very strong; else only inclosing a part of it.

When it is probable that an army will relieve the place, a line of circumvallation is to be made further into the campaign, this line is to be carried round about, inclosing ground enough for the besiegers to lodge in; its parapet is thrown up towards the campaign; and at proper distances, forts and redoubts to be placed, to defend the soldiers and workmen.

The next thing is *opening the trenches*, or beginning the *lines of approach*, which are to be carried on obliquely towards the town, so as not be exposed by any of the outworks.

As soon as the besiegers approach towards the town, the besieged are to shoot off several cannon for a signal that all may come into the town. A few several shot are to be made to keep off light horsemen and spies.

The besiegers will endeavour to stop all ways, Fig. 1. passages, and communications of the town and country, to hinder any succours or any intelligence that may be sent; this is *blocking up* the town.

While the trenches are drawing near the town; the garrison ought to make a sally every opportunity, to annoy them in their works; this will probably draw the army nearer to the town, which may be cannonaded from the town, and perhaps some of the principal officers cut off.

When the garrison is strong, and the enemy attacks any out-works, makes any mines or the like; the garrison is to *counter-mine* them, and obstinately defend them, and to make *retrenchments* and *barriers* all along, and lose all things inch by inch.

As the trenches go on, batteries must be erected against sallies, and when they come nearer the town, sallyes must be made to ruin the defences. At the same time the cannon from the town must play upon these batteries of the enemy to demolish them. The besieged constantly do this, the besiegers may be forced to raise the siege.

The besiegers, while they work in the trenches, must be supplied with faggots, sacks of earth, gabions, &c. for defence. And to avoid musket-shot, these trenches must be carried on in the night, and are carried on by turnings and windings, from right to left, but never to go directly towards any work of the place, for it would be impossible to stay in the trenches. And near the trenches, bodies of soldiers must be posted to defend the pioneers.

When the enemy is come to the glacis, against the efforts of the town. They must dig through the *counterscarp*, or work by *sap*, till they come to the ditch. A battery then must be raised, to facilitate the passing of the ditch; the next thing is to make a gallery, which must be defended by the batteries. In the mean time the cannon from the town

Fig. will endeavour to destroy the enemy's battery ; and if it does, it retards the work, which cannot go forward, till the battery be repaired or another erected. If the enemy be successful in getting through the ditch, working in the night especially, they want to blow up the bastion, the miners must be set to work to make a mine ; and as soon as it is sprung, the besiegers go and give the assault. But if the besieged by help of their counter-mines, give air to it, it will have no effect.

If making a mine be impracticable, the besiegers batteries must continually play upon the flanks of the fortification, to dismount their cannon ; while other batteries are employed to make a breach, or to break the faces of the bastion, which being done, the enemy must mount the breach, or scale the place, and take it by storm. If the besieged are out of hopes of defending the place, they must either surrender, or capitulate.

After this manner the whole time of the siege is employed in erecting works of defence, and raising batteries, to fire upon and destroy the fortifications. While on the contrary, the time of the besieged is taken up in destroying the others works and driving them back. And this must continue till the place is either taken or the siege raised.

Thus I have given a short account of the manner of proceeding in a siege ; but my room will not permit to make a longer discourse.

A R T. XIII.

G U N N E R Y.

GUNNERY is an art that teaches how to shoot with great guns, by finding what elevation any gun ought to have to hit a mark whose position is given; as likewise what quantity of powder is requisite for that purpose. It teaches the method of managing all sorts of artillery, in attacking an enemy, or in defending or attacking a town. It is a proper sequel to fortification, and together make but one compleat art, which is the art of war. This art of gunnery depends on the motion of projectiles, which gives rules for finding the velocities, distances, elevations, &c. of all requisites concerned in shooting, so far as the resistance of the air is supposed to be nothing, or very small; for when the resistance of the air is considered, it makes the Problems of Gunnery too complex to be of any use. When the resistance of the air is so great that the rules for projectiles do not hold, then recourse must be had to tables made from experiments.

D E F I N I T I O N S.

D E F. I.

Range or amplitude, is the distance to which a shot is shot, from a gun, which is measured upon the plane of projection.

D E F. II.

Elevation, is the angle to which the axis of the barrel of the gun is raised above the plane of projection.

T 3

D E F.

G U N N E R Y.

D E F. III.

Altitude, is the height a ball acquires, above the plane of projection, or the highest point of its tract.

D E F. IV.

Battery, a place to plant guns on. It is laid with planks and sleepers, to rest on; that the wheels of the carriages may not sink: the guns are generally placed about 12 or 14 feet from one another.

D E F. V.

Charge, is the quantity of powder and ball proper for any gun.

D E F. VI.

Cannon or great guns, are round hollow cylinders made either of brass or iron, charged with powder and ball to shoot with, and mounted upon carriages to be easily moved from one place to another. There are several sizes of them, called by several names, as whole cannon, bastard cannon, demi-cannon, or 24 pounders, whole culverin or 16 pounders, demi-culverin 6 pounders, sakers, minions 3 pounders, drakes and pedereroes, Cannon royal, the bore is 8 inches diameter, 12 feet long; charge 32 pounds of powder, ball 12 inches diameter, or 48 pounds weight. There are several particular terms used in Gunnery, as

To tertiate a gun, is to try whether a cannon has a due thickness of metal in every part.

To dispart a cannon, is to set a mark on the muzzle-ring to be of an equal height as the base-ring, so that a line drawn between them, shall be parallel to the axis. This mark is for the gunner to take aim by.

Point blank, is the distance a ball can be shot Fig.

in a gun, without sinking below the line of direction; or how far it will go in a direct line.

The *windage* of a gun, is the difference between diameter of the ball and the diameter of the bore.

The parts of a piece of ordnance or cannon, are, the *caliber*, which is the diameter of the bore.

Chamber, the part where the powder and shot lie.

Trunnion-ring, the ring going round the gun at the breech.

Muzzle-ring, the ring at the muzzle.

Wedges, the two knobs or pins that hold the gun in the carriage.

Things belonging to a cannon are, ladles, sponges, bridges; also needles, thread, twine, nails, hand-axes, crows of iron, baskets, the pick-ax, mat- shovel, spade, &c. to work with.

These things are to be shot out of a cannon.

Bombs, which are hollow balls of iron; these are charged with powder; and a *fuzee* (which is a composition that burns slowly) put in at the touch-hole, to last all the time the bomb is flying, and when it falls, sets the powder in fire, which bursts the bomb.

Granadoes, are small bombs or shells of iron; sometimes made of wood or tin, or even of paste-board. These are thrown by hand amongst men.

Red hot bullets, these are cannon-balls heated till they have a flame heat, and shot out of a cannon, to give light to the field, or to fire a town.

Flaming barrels, or fire barrels, called also *thundering barrels*. These are filled with granadoes and pots, or pots filled with gun-powder and a granado; these barrels are shot out of a mortar-ice, which is a very short gun, with an extraordinary wide bore.

Carcasses, are boxes full of powder, made with hoops, filled with granadoes and pistol barrels charged, wrapt up in oiled tow, or other combustible

Fig. tible matter, and shot out of a mortar. The mortars are elevated pretty high, that the bombs &c. may fall from a great height.

Pederero or *petterero*, a small cannon to fire stones or pieces of old iron out of, when the enemy attempts to board a ship, &c.

Petard, an engine to shoot with, in form of high crowned hat; about 8 inches deep, and diameter at the mouth, and 1 inch or 2 at the breech. It is filled with powder, and covered with a plank bound down with ropes, used to break down gates, bridges, &c.

Sausages, are pipes made of tarred cloth, filled with powder, and rolled up in form of a gut, about 2 inches diameter, it is to burn gradually, and reaches from the mine to the engineer, and is set on fire when the mine is to be blown up.

Ammunition, signifies all sorts of warlike stores particularly powder and ball.

Artillery, all sorts of ordnance, or guns great and small, as cannons, mortars, muskets, &c. and fire arms.

Blunderbuss, a short gun with a very wide bore to carry a number of bullets, to shoot in a crowd.

A *fuzee*, a pipe full of wild-fire put into the touch-hole of a bomb to set fire to it.

Shot, all sorts of bullets for fire arms, from the cannon to the pistol. Those for cannon are iron; those for muskets, carbines, and pistols, lead. *Chain shot* and *bar-shot*, are two half bullets joined by a chain or iron bar, which cut every thing they meet with.

Obuse guns, are those guns in a ship that point directly forward or backward.

Swivel guns, guns that move in an iron frame that goes into the ship's side.

Fire-ball, this is made of powder and other combustible, as big as a granado, wrapt up in

and covered with paper; when primed and set on fire, it is cast into any works in the night to discover them.

To nail up cannon, is to drive a large nail into the touch-hole, or pieces of stones; this is to render them useless to the enemy, when they cannot be carried off. The remedy for this is, to drill a new touch-hole.

P R O P. I.

To describe the composition, the nature and force of gun-powder.

Gun-powder is but a late invention, said to be found out about the year 1340, or not much sooner. Before this invention men fought with bows and arrows, and such like things, which did not do a deal of execution. But the dreadful havock which is made by shooting balls of iron and lead out of guns of all sizes, by the force of gun-powder, caused all the former sort to be laid aside.

Gun-powder is a mixture of charcoal, brimstone, and nitre; the proportion of these is different in different authors. But powder made for the government is composed of 10 pounds of brimstone, 15 pounds of charcoal, and 75 pounds of nitre. Powder made for the use of merchants has less nitre and more charcoal. At first powder was made fine like meal, but afterwards made into corns, which was found to fire quicker than the meal powder. To corn it, it must be put through two or three sieves. The powder is to be moistened and reduced to a sort of paste, and put into the first sieve, which has large holes, it is broke in pieces by a piece of wood, and passes through the sieve while it is working. The next sieve which is smaller retains the corns, but lets the dust go thro'.

The

Fig. The corns passed through this second sieve are stopped at the third sieve, with still smaller holes. The powder which only goes through one sieve is used by great guns, and called cannon powder. The smaller sort, which passes through two sieves, is used by small guns, and called pistol powder. To glaze the powder, or make it shine, they shake it in a barrel that has had black lead in it.

The sudden fulmination of nitre with sulphur is the principle upon which the action of gun-powder is founded. Sulphur thrown upon melted nitre causes an inflammation and an explosion. The charcoal is only used to set the mixture suddenly on fire.

There is a mixture called *pulvis fulminans*, whose explosion is greater than gun-powder, but it does not fire so quickly. It is made thus, 1 part of brimstone, 2 parts of salt of tartar, 3 parts of saltpetre; grind them very small, and mix them together well.

The sudden firing of gun-powder is owing to the nitre, which prodigiously increases the inflammability of combustible substances. And the vehement explosion arises from the great quantity of air contained in the gun-powder into which it is converted by firing. But if gun-powder be beat small, it will not explode at once, but burn gradually; and therefore in all fire-works, where the powder is to burn gradually, it is beat into dust.

That gun-powder when fired produces a permanent elastic fluid, has been proved by many experiments. And the force of fired gun-powder is the elasticity or pressure of the fluid produced by firing it, and is the same as the elastic force of so much air compressed into the same space with the gun-powder. Likewise the elastic force of this fiery fluid is proportional to its density, as is proved by many experiments. And further the elasticity

If this fluid increases by heat, and diminishes by Fig. cold, like the air; also its weight, *ceteris paribus*, equal to the weight of an equal bulk of air. If a grain of common powder be fired, the fluid produced by it will be 230 or 240 times its bulk, and so for a greater quantity. And therefore if this fluid was confined to the space of the gun-powder, it would be 230 or 240 times as strong as common air; that is, its elastic force would be 230 or 240 times as strong as the pressure of the atmosphere; and that without considering any augmentation of that force by heat. But if air be heated to the degree of red hot iron, its elasticity will be above 4 times as great at that of common air. And therefore the motive elastic force of this heated rarified fluid, produced by the firing of gun-powder is 1000 times as great as the pressure of the atmosphere.

It has been disputed, whether gun-powder fires instantaneously, or requires time for the flame to run from one part to another. To decide this question, the royal society made several experiments, the result whereof was, that out of a very short barrel 5 or 6 inches long, a quantity of powder was blown out unfired, amounting sometimes to $\frac{1}{4}$ the quantity of the charge. But if she was charged with a ball, the quantity was less, being $\frac{1}{2}$ or $\frac{1}{3}$ of the charge, and this was owing to the time being prolonged by the weight of the ball. But when a gun of the length of 3 or 4 feet was fired, hardly $\frac{1}{10}$ part of the powder was unfired; and when loaded with a ball, the quantity was far less, and for the most part so inconsiderable as to deserve no notice.

It is evident, being confirmed by all experience, that dry powder fires quicker than damp; and is consequently so much the stronger. And powder may be so dry, as that the whole may fire before the bullet gets out of the gun; or even before it
be

Fig. be sensibly moved. Some experiments have been made, which shew, that dry powder will shoot a ball twice as far as the same powder used when it is damp.

P R O P. II.

To find the velocity of a ball at the mouth of a gun, supposing all the powder to fire instantaneously.

54. Let AD be the hollow cylinder of the gun, AB = a the gun-powder, the length AD = l , the variable length AC = x , d = diameter of the bore, $c = 3.1416$, $b = 16.1$, w = weight of the ball, p = pressure of the atmosphere on 1 inch, f = force of the gun-powder at B, v = velocity of the ball at C; then $\frac{cdd}{4}$ = area of the section, and

$\frac{cddp}{4}$ = pressure of the atmosphere on the bullet.

Then $f = \frac{1000cddp}{4}$, by the last Prop. And $x : a$

$:: f : \frac{af}{x}$ = force on the bullet at C. Then (by

Cor. Prob. I. Sect. III. Fluxions,) $v\dot{v}$ is as $\frac{F\dot{x}}{b}$.

And in the case of falling bodies, $v = 2b\dot{x}$, $\dot{v} = \dot{x}$,

and $F = w$, $\dot{s} = \dot{x}$; therefore $2b\dot{x} : \frac{w\dot{x}}{b} :: v\dot{v} :$

$\frac{af}{x} \times \frac{\dot{x}}{b}$, or $2b : w :: v\dot{v} : \frac{af\dot{x}}{x}$; therefore $v\dot{v} =$

$\frac{2baf\dot{x}}{wx}$, and the fluent is $\frac{v^2}{2} = \frac{2baf}{w} \times b \cdot \log. x$.

But in B, $x = a$; therefore $\frac{vv}{2} = \frac{2baf}{w} \times b \cdot \log. \frac{x}{a}$.

, and when x comes to be equal to AD, $\frac{vv}{2} = \text{Fig. 54.}$

$$\frac{baf}{w} \times b. \log. \frac{l}{a}; \text{ therefore } vv = \frac{4baf}{w} \times b. \log.$$

$$l, \text{ and } v = \sqrt{\frac{4baf}{w} \times b. l. \frac{l}{a}} =$$

$$\sqrt{\frac{4ba}{w} \times \frac{1000cddp}{4} \times b. \log. \frac{l}{a}} =$$

$$\sqrt{\frac{1000cbaddp}{w} \times \text{hyp. log. } \frac{l}{a}}.$$

Cor. In all guns, the square of the velocity is as

$$\frac{l}{a} \times \log. \frac{l}{a}, \text{ or } \frac{a}{d} \times \log. \frac{l}{a}.$$

E X A M. I.

Let the length of the gun be 3 feet, diameter of a leaden ball 1 inch, a the charge of powder 2

ches, then $w = .214$, $p = 14.7$, and hyp. log.

or 18 = 2.890, then

log. 1000	3.00000
log. $c = 3.1416$	0.49714
log. $b = 16.1$	1.20682
log. $add = .1666$ —	1.22182
log. $p = 14.7$	1.16732
log. 2.890	0.46089
	5.55399
sub. log. $w = .214 =$	1.33041
	6.22358
half —	3.11179
velocity —	1296 feet in a second.

E X A M. 2.

Let the length of a gun be 9 feet, the diameter of the bore, or rather of the iron shot 4 inches,

its

Fig. its weight w 9 pounds, $a = 8$ inches the charge of
 54 powder, hyp. log. $\frac{l}{a}$ or $13.5 = 2.602$,

log. 1000	3.00000
log. c , 3.1416	0.49714
log. b , 16.1	1.20682
log. p , 14.7	1.16732
log. add , 10.66	1.03775
log. 2.602	0.41530
	7.31433
log. w , 9	0.95424
	6.36009
half	3.18004
velocity	1514 feet in a second

P R O P. III.

To find the velocity of a ball at the mouth of a cannon or any gun; supposing the powder to fire gradually so as to be all fired when the ball is at the mouth of the gun.

We must first find the relation between the time and space described. The force at any instant will be as the quantity of powder fired, divided by the space it is contained in; therefore putting the quantities as in the last Prop. we shall have the force $\propto \frac{t}{x}$. Now suppose $x \propto t^n$, to find n . By Prob

I. Sect. III. Fluxions, $\dot{v}v \propto F\dot{x}$, and $F \propto$

$$\propto \frac{x^{\frac{1}{n}}}{x} \propto \frac{1}{x^{\frac{n-1}{n}}}, \text{ therefore } \dot{v}v \propto \frac{\dot{x}}{x^{\frac{n-1}{n}}}, \text{ and } v \propto x^{\frac{1}{n}}$$

the fluent $v^1 \propto nx^{\frac{1}{n}}$. Also $\dot{x} \propto v^1 \propto v \times \frac{1}{n}$ 54.

$\dot{x} \propto x^{\frac{1}{2n}} \times x^{\frac{1}{n}-1} \dot{x} \propto x^{\frac{3-2n}{2n}} \dot{x}$, and the

fluent $\propto x^{\frac{3}{2n}}$, or $x^1 \propto x^{\frac{3}{2n}}$, whence equating

the indices, $1 = \frac{3}{2n}$, and $2n = 3$, and $n = \frac{3}{2}$.

Therefore $x \propto t^{\frac{2}{3}}$. This being known, put $g =$ whole time of passing through AD.

We shall now have $f = \frac{1000cd\dot{p}}{4}$ as in the last

prop. for the force at the first firing, and $\frac{af}{l} =$

force at D when it is all fired. And to find the

force at C; since it is generally as $\frac{t}{x}$, we shall

have $\frac{g}{l} \left(\frac{\text{time}}{\text{space}} \right) : \frac{af}{l} \left(\text{force at D} \right) :: \frac{t}{x} \left(\frac{\text{time}}{\text{space}} \right) :$

force at C $= \frac{aft}{gx}$. Then since $v\dot{v} \propto F\dot{s}$, the bo-

dy being given; and in a falling body $v = 2b$,

$\dot{v} = \dot{x}$, $F = w$, $\dot{s} = \dot{x}$; therefore $2b\dot{x} : w\dot{x} ::$

$\frac{aft\dot{x}}{gx}$. Therefore $v\dot{v} = \frac{2baf\dot{x}}{gx}$. But $l : g^{\frac{2}{3}}$

$x : t^{\frac{2}{3}} = \frac{g^{\frac{2}{3}}x}{l}$, and $t = \frac{gx^{\frac{3}{2}}}{l^{\frac{2}{3}}}$; therefore $v\dot{v} =$

$\frac{af}{x} \times \frac{gx^{\frac{3}{2}}\dot{x}}{l^{\frac{2}{3}}} = \frac{2bafx^{-\frac{1}{2}}\dot{x}}{wl^{\frac{2}{3}}}$. And the fluent is

$\frac{2bafx^{\frac{1}{2}}}{\frac{2}{3}wl^{\frac{2}{3}}}$, or $vv = \frac{6bafx^{\frac{3}{2}}}{wl^{\frac{2}{3}}}$, and $v = \sqrt{\frac{6bafx^{\frac{3}{2}}}{wl^{\frac{2}{3}}}}$.

and when $x = l$, then $v = \sqrt{\frac{6baf}{w}} =$

$\sqrt{6ba}$

Fig. 54. $\sqrt{\frac{6ba}{w}} \times \frac{1000cddp}{4} = V = \sqrt{\frac{\frac{3}{2} \times 1000cbaddp}{w}}$

Several Corollaries follow from this solution, as

Cor. 1. *The last velocity at D will be the same whatever length the gun be of; provided the height of the charge be given, and the powder be of such goodness as to be all fired, just when the ball reaches D*

For l does not come into the computation, and remaining the same, V will remain the same.

Cor. 2. *The powder being the same, the final velocity will be as the square root of the height of the charge.*

For the rest remaining, the velocity will be $\sqrt{a}$.

Cor. 3. *If l be the length of gun, wherein the charge of powder will all be fired at the muzzle D. Then in any shorter gun AC, the same powder and charge remaining, the ultimate velocity will be*

$$\sqrt[3]{\frac{x}{l}}$$

For the rest remaining, V is as $\sqrt{\frac{x^{\frac{2}{3}}}{l^{\frac{2}{3}}}}$.

Cor. 4. *And in this last case, the velocity with the charge a will be no greater than with the charge*

$$\frac{x^{\frac{2}{3}}}{l^{\frac{2}{3}}} a \text{ or } a \sqrt[3]{\frac{xx}{ll}}$$

For let y = the charge that will just be fired at D. Then by supposition $a : y :: g : t :: l^{\frac{2}{3}} : x^{\frac{2}{3}}$, when $y = \frac{ax^{\frac{2}{3}}}{l^{\frac{2}{3}}}$; and the superfluous powder does nothing

Cor. 5. *Supposing as in Cor. 3. and in any gun longer than l , the velocity at last will also be as*

$$\sqrt[3]{\frac{x}{l}}$$

Cor. 6. In guns of different bores, the rest remain- 54
as before, the velocity at the muzzle will be as

$$\frac{a}{d}$$

For it is as $\sqrt{\frac{dda}{w}}$ or $\sqrt{\frac{dda}{d^3}}$ or $\sqrt{\frac{a}{d}}$

E X A M. 1.

Let the gun be 3 feet long, $d = 1$ inch, the dia-
ter of the ball of lead, $a = 2$ inches of powder,
 $w = .214$, $p = 14.7$.

$$\log. 1000 \quad 3.00000$$

$$\log. c, 3.1416 \quad 0.49714$$

$$\log. b, 16.1 \quad 1.20682$$

$$\log. add = .1666 \quad - \quad 1.22182$$

$$\log. 1.5 \quad 0.17609$$

$$\log. p \ 14.7 \quad 1.16732$$

$$\underline{\hspace{1cm}} 5.26919$$

$$\log. w \ .214 \quad - \quad 1.33041$$

$$\underline{\hspace{1cm}} 5.93878$$

$$\text{half} \quad 2.96939$$

$$\text{velocity at D} = 932 \text{ feet per second.}$$

E X A M. 2.

If the length of a gun be 9 feet, diameter of her
being iron = 4 inches, weight $w = 9$ lb. $a = 8$
is the charge.

U

log.

Fig.	log. 1000	3.00000
54.	log. c , 3.1416	0.49714
	log. b , 16.1	1.20682
	log. p , 14.7	1.16732
	log. add , 10.66	1.02775
	log. 1.5	0.17609
		7.07512
	log. w , 9	0.95424
		6.12088
	half	3.06044
	velocity =	1149 feet per second

PROP. IV.

Supposing the powder in a gun to fire instantaneously to find the depth of the charge, to produce the greatest velocity possible.

It is shewn in Prop. II. that the velocity in a supposition is $\sqrt{\frac{1000cbaddp}{w}} \times b \log. \frac{l}{a}$

supposing all things given except the charge then $a \log. \frac{l}{a}$ will be a maximum, put x for

variable quantity, then $x \log. \frac{l}{x} = \text{maximum}$

$y = b \log. \frac{l}{x}$, then $xy = \text{max.}$ and $xy + y\dot{x}$

The fluxion of $\frac{l}{x}$ is $\frac{-l\dot{x}}{xx}$, and this divided

$\frac{l}{x}$; gives $\frac{-l\dot{x}}{xx} \times \frac{x}{l} = \frac{-\dot{x}}{x}$, therefore

$\frac{-\dot{x}}{x}$; and $xy + y\dot{x} = x \times \frac{-\dot{x}}{x} + y\dot{x} = 0$, or

$y\dot{x} = 0$, and $y = 1 = b$. $\log. \frac{l}{x}$; that is, Fig. 54

$30258 \times \text{com. log. } \frac{l}{x} = 1$, and $\text{com. log. } \frac{l}{x}$

$.4343$; and $\frac{l}{x} = \text{num. of log. } .4343 = 2.718$;

therefore $2.718 x = l$, and $x = \frac{l}{2.718}$, or $x =$

$.679 l$, when the velocity generated is a maximum.

S C H O L.

This can be of no use in any gun, except they be exceeding short; for such a charge will burst any gun, of a moderate length, to pieces.

P R O P. V.

To find the due quantity of powder proper for the charge of any gun.

This must be found by experiments for some particular gun. Take any gun, and a ball proper for it, and first let it be charged with a quantity of powder equal to $\frac{1}{3}$ the weight of the ball, and elevating the piece to any given angle, suppose 45 degrees, let it be discharged, and let the random distance of the shot be measured. Again, let it be charged with powder equal to half the weight of the shot, and discharged, and the random measured. And lastly, let the charge of powder be equal to the weight of the shot, and the random also measured. Then the several distances or randoms compared, will shew which is the most convenient charge. For though the greatest charge of powder shoots the furthest, yet when the increase of distance is but small; it is better to take up with the lesser charge, because so much powder is saved.

U 2

But

Fig. But there are so many irregularities attends this practice, that it is proper to repeat the experiment over and over; and lastly, to take a mean for each charge. For in the same piece, with the same charge, and placed at the same angle of elevation there is such difference in the distances, that one cannot but be surprized at it. And therefore it is necessary that several experiments be made with the same charge. By comparing many experiments together, I take that to be the most proper charge when the weight of the powder is half the weight of the ball.

The proper charge for one gun being known, the charge for any other gun is known by this rule.

Take the charges of powder in proportion to the cubes of the diameters of the balls, or as the cubes of the diameters of the bores of the guns.

E X A M.

If a ball 4 inches diameter requires $4\frac{1}{2}$ lb. powder; how much powder will a ball 6 inches diameter require?

$$\begin{array}{r}
 4 \\
 \underline{4} \\
 16 \\
 \underline{4} \\
 64 : 4\frac{1}{2} :: 216 \\
 \quad \quad \quad 4\frac{1}{2} \\
 \quad \quad \quad \underline{864} \\
 \quad \quad \quad 108 \\
 \quad \quad \quad \underline{64} \\
 64)972(15.2 \text{ lb. answer.} \\
 \quad \underline{64} \\
 \quad 332 \\
 \quad \underline{320} \\
 \quad 12
 \end{array}$$

Some people say, that guns ought to be loaded Fig.

according to their weight; and give it as a rule, that a quarter of a pound of powder should be allowed for every hundred weight of the gun, if it be of iron; but something more for a brass gun.

But the quantity of powder ought to be so proportioned to the ball, as the business to be done requires, or according to the execution to be done. And modern engineers, that have had a deal of experience, tell us, that the charge of powder ought not to exceed $\frac{1}{3}$ the weight of the ball. For though greater velocities are generated by greater charges, yet by the great resistance of the air, these greater velocities are soon reduced to the velocities generated by the lesser charges.

But guns may be charged by measure as well as by weight. For a pound of powder fills 31.06 cubical inches; therefore say, as 1 $\frac{1}{2}$ to 31.06 inches; so half, or a third, of the weight of the ball, to the charge in cubical inches.

It must be observed, that there is great difference in the goodness of powder; and even the same powder will be stronger or weaker, according as it is kept dry or moist; for powder will imbibe moisture, and that abates the strength of it. And if powder be known to be weaker than ordinary, more must be taken for a charge.

Note. powder for proof, (that is, for trying the strength and goodness of guns) must be double the quantity of powder for service, as some say; and others say, as 3 to 2.

When a gun is to be charged with a *cartridge* (which is a bag, or roll of canvass, paper or parchment, of the bigness of the gun's bore, to hold a charge); having the weight of a charge of powder in pounds, multiply it by 31.06, and call the product p . Let d = diameter of the bore in inches.

U 3

Fig. inches. Then $\frac{p}{.7854dd} = \text{length of the carriage.}$

Some authors tell us, the charge of powder for great guns must be 2 diameters of the ball; and for lesser guns 3 diameters.

P R O P. VI.

To describe the construction and use of a great gun.

A gun is said to be well fortified, when there is sufficient thickness of the metal at the breech, and in every other place. For if a gun be badly fortified, or too thin of metal, it will endanger bursting, and in such, the charge ought to be moderate. When a gun is well fortified, the diameter of the bore must be to the diameter of the gun at the breech, as 2 to 7. And then the diameter of the bore being 2, the thickness of the metal will be $\frac{5}{2}$ or $2\frac{1}{2}$. And therefore the diameter of the bore will be to the thickness of the metal as $1\frac{1}{2}$, or as 4 to 5.

Hence the circumference of the gun at the touch-hole, is 11 diameters of the bore. For if the diameter of the bore be 2, the diameter of the gun then is 7, and the circumference 22, and 22 to 11 as 2 to 1. The circumference at the touch-hole being 11, the circumference at the trunnions must be 9, and at the muzzle 7. And this is esteemed the true fortification for an iron gun.

But for brass guns, the circumference at the breech must be 9 diameters of the bore, 7 at the trunnions, and 5 at the muzzle. Therefore the thickness of the metal at the breech, is about the diameter of the bore.

And the length of either sort is to be 7 diameters of the gun at the breech, though this is

material for the length, which upon occasion Fig. may be different.

But the rule here laid down for a gun truly forged seems only to be proper for one sort. For the thickness of the metal is estimated the strength of the gun, and ought to be proportional to the force within; but that force is as the quantity of powder, and that is as the square of the diameter of the bore. Therefore in wider pieces, the metal ought to be thicker than the rule makes it. For if the bore of an iron gun be 6 inches, and $7\frac{1}{2}$ inches (by the rule above) be the proper thickness of the metal. Then for a bore of 8 inches, 10 inches should be the thickness of the metal, which is too little. For it should be as $36 : 64 :: 7\frac{1}{2} : 13\frac{1}{2}$ inches for the true thickness of the metal, for 8 inches bore.

But if the gun be thinner on one side than the other, her strength must be estimated by the thinnest side. Or if there be flaws or places eaten away with rust, called *honey-combs*, these weaken a gun very much, and if they be deep, such guns ought not to be used.

To adjust the ball to the piece, you must use such a ball, that the difference of the diameters of the ball and bore may be about $\frac{1}{32}$ the diameter of the ball; and this ball will suit the gun. If the difference be less than that, the ball (being of iron) is in danger of sticking, and the gun bursting. And if it be more, a deal of the strength of the powder will be lost, in passing by the ball. This allowance is called the *windage*.

How great guns are to be managed in shooting. First the gun being placed upon her carriage, she must be charged with powder; to do which the gunner makes use of a *ladle*, which is a long staff with a tin box at the end, which being filled once,

Fig. or oftener, with powder taken out of the *budge barrel*, is a just charge for the gun.

Next the wad is to be put in, and rammed down with the *rammer*, which is a long pole on purpose.

Then the shot must be taken and tumbled in the gun, going close to the wad.

Then by help of the quadrant and a straight rule laid to the inside of the bore, the piece must be elevated to a proper angle.

Then by help of a *priming iron*, which is a long small piece of iron, a hole is made through the cartridge to the powder, and the powder is to be poured into the touch-hole to the top, and covered with a thin plate of lead, called the *apron*.

And when she is to be fired, a hot iron is applied to the touch-hole, which sets the powder on fire by which she is discharged.

Then the gun is to be scoured with a sponge that no fire remain in her. A *sponge* is a long stick with a roll at one end, covered with a sheep skin of the bigness of the gun's bore. Great guns must be cooled leisurely with water after every shot; being much heated by often shooting, they ender bursting.

The platform of a battery ought to be firm; if it shake when a gun is fired, the shot will be very uncertain.

In every battery, there must be half as many spare guns, as there are mounted; to supply the that are disabled.

It is a very important improvement in gunnery called by the French, shooting by *ricochet*. Which is, to shoot with small charges of powder, and elevating the gun, so that the shot may just go over the parapet, and drop into their works. By this means more mischief is done than in firing directly against the works.

Some authors have given Tables of Gunnery, Fig. shewing the name, weight, length, diameter of the bore, weight of shot and powder, &c. of which the following is one, for all sorts of ordnance.

name.	weight gun.	length gun.	diam. bore	weight shot.	weight powder.	point blank.
	Cwt.	feet.	inch.	poun.	poun.	feet.
cannon 8	70	12	8	60	30	900
cannon 7	$60\frac{1}{2}$	11	7	42	21	900
semi-cann.	$50\frac{1}{2}$	10	$6\frac{1}{2}$	32	16	900
culverin	40	12	$5\frac{1}{3}$	18	10	900
semi-culv.	30	10	$4\frac{1}{2}$	11	8	890
saker	20	9	$3\frac{3}{4}$	6	4	800
minion	12	7	3	$3\frac{1}{4}$	$2\frac{1}{2}$	600
falcon	6	7	$2\frac{3}{4}$	$2\frac{1}{2}$	$2\frac{1}{4}$	600
falconet	$3\frac{1}{2}$	6	$2\frac{1}{4}$	$1\frac{1}{3}$	$1\frac{1}{4}$	520
rabinet	$2\frac{1}{2}$	$5\frac{1}{2}$	$1\frac{1}{2}$	$0\frac{1}{2}$	$0\frac{3}{4}$	500
base	$1\frac{3}{4}$	$4\frac{1}{2}$	$1\frac{1}{4}$	$0\frac{1}{2}$	$0\frac{1}{2}$	400

P R O P. VII.

To find how much powder will fill any bomb or graduated shell.

R U L E.

Cube the inner diameter of the shell, and divide by 59.3, the quotient will be the pounds of powder that will fill the shell.

For the shell is in form of a globe or sphere, let

d = the diameter; then $\frac{11d^3}{21}$ = solidity in inches,

and therefore $\frac{11d^3}{21 \times 31.06}$, that is, $\frac{d^3}{59.3}$ shews the pounds of powder.

In a shell the lowest part is the thickest and heaviest, and therefore will fly foremost, and it will fall

Fig. fall upon that side, and not on the fuzee, which would put it out.

E X A M.

Let the diameter of the shell be 13 inches,

13	59.3)21.97(37.05 lb.
13	: : 17 79 the answer.
39	4180
13	
169	29
13	
507	
169	
2197	

P R O P. VIII.

Having given the diameter of an iron or leaden ball to find the weight of it ; and the contrary.

R U L E.

As 100 to the cube of the diameter in inches so is 14 to the weight in pounds, for the iron ball And the leaden ball will weigh half as much more

E X A M. I.

Suppose the diameter of the shot to be 6 inches

6	100 : 14 :: 216
6	14
36	864
6	
	216
216	
	100)3024(30.24 lb. for iron
	15.12
	45.36 for lead.

E X A M

E X A M. 2.

If an iron ball weigh 48*lb*. what is the diameter ?

$48 :: 100 :$

$14)4800(342.8$, whose cube root is 7,

42

60

56

40

28

120

120

so the diameter is 7 inches.

Note, the weight of a ball of stone is $\frac{3}{4}$ of the weight of an iron ball.

P R O P. IX.

To find the distance to any fort or battery, by the ringing of a gun there.

Sound is known to move 1140 feet in a second of time, but the motion of light is instantaneous. Therefore observe with a pendulum how many seconds there are between the flash of fire and the report of the gun; this number of seconds multiplied by 1140, gives the number of feet for the distance from the place where the gun is fired.

Or thus, count how many double beats of a common watch passes between the firing of the gun, and the sound reaching your ear; multiply this by 490, gives the distance in feet.

Or thus, if your pulse be regular, count the number of pulses between the flash and the report; which multiply by 970, gives the distance in feet.

E X A M.

E X A M. 1.

I observed a gun fired at a certain place, and observation found, that seven double beats (or single ones) of my watch passed between the firing and the sound, what is the distance ?

$$\begin{array}{r} 490 \\ 7 \\ \hline 3430 \end{array} \text{ feet the distance.}$$

E X A M. 2.

Another person counted the number of pulses
be $3\frac{1}{2}$.

$$\begin{array}{r} 970 \\ 3\frac{1}{2} \\ \hline 2910 \\ 485 \\ \hline 3395 \end{array} \text{ feet for the distance.}$$

P R O P. X.

Besides the resistance of the air, there is another irregular force by which a projectile is made to deflect from a direct course. To explain this deflection, and cure it.

It is hardly possible for a ball to be shot out of a gun, without its rubbing against one side or another of the barrel; and the friction it receives by that means, gives it a whirling motion round an axis, which is always perpendicular to the axis of the barrel or to the tract of the ball. The consequence of this is, that one side of the ball meets with a greater resistance of the air than the other side; and the air acting obliquely against that side with the greater resistance, will force it to move towards

wards that side where is the least resistance; and Fig. quantity of this deviation will be as the difference of the resistance of one side above the other. and consequently the ball will always deflect towards that side of the barrel where the friction happened; for that side of the ball being retarded in motion, meets with the least resistance. But it is impossible to know before-hand, on which side of the barrel the friction will happen. But when the shot is over one may nearly determine on which side it was. For if the shot be over the mark, it is on the upper side; if short, on the under one. on the right or left, it is on the right or left respectively. And besides, this tract of deviation must be a curve line. For as this disturbing force is continual; every succeeding part of the shot will deviate (the same way) from the former; which is the nature of a curve line.

I have been long acquainted with this irregularity and its effects, which I found by experience when I used to practice shooting; and it presently occurred to me, that the greater resistance on one side of the ball, was the true cause of its going out of the line of direction. And to satisfy myself about it, I suspended a wooden ball loaded with powder, in a string, and tied it to a tree that hung over a river, that it might play freely in the stream, and noting the place where it rested, I then twisted the string, by turning the globe often about, and putting it into its former place, it rested but a little while; for as the string began to untwist, it moved gradually towards the side which conspired with the motion of the water. And being at its furthest extent, it rested till the motion began to diminish, and then it came gradually back to its first place, and rested there till the motion of the globe twisted the string the contrary way; and then it moved to the other side. And thus it made several

Fig. several vibrations to and fro, till the motion is spent, and then it rested in the first place. This several times repeated.

I also tried the same in a strong wind, with like success; for the ball always deviated from the plane of the winds motion towards that hand where it was least resisted.

Now to remedy this reflection, one way is, use bullets that are not round, but oblong, something like a slug. But then they ought to be turned in a lathe or throw; that the fore end may be regular, and all sides alike; that the air may strike equally on all sides. Such a body as this, shot out of a gun, cannot by friction be made to revolve about an axis, and therefore that irregular force and its effect will be prevented.

Another way to prevent this deflection is to make the guns rifle bored; these rifled barrels are made with several threads of a screw running spiral way on the inside of the barrel; between these threads are channels cut in the bore, all which must be exactly parallel to one another, and make about one revolution in the length of the barrel, going uniformly about. The number of these threads is different, according to the wideness of the barrel.

There are different methods of charging the pieces, one is this. After the powder is put in, they take a bullet something bigger than the bore of the gun, and grease it well, and putting it in the mouth of the piece, they ram it down with an iron rammer, hollow at the end; in ramming down the bullet the spiral threads enter and cut into the bullet, and cause it to turn round in going down, and being shot out, it follows the same direction of the rifles; which causes it to turn round on an axis parallel to the gun's bore.

Another way is to charge them at the breech where there is a hole to put in the powder and ball.

and then a screw screws in to fill up the hole. But Fig.

Some barrels screw off at the breech, to be charged.

These guns are made stronger at the breech than common; and it is plain they can only be used for lead bullets, for iron will receive no impression.

And thus a bullet shot out of a rifled barrel, besides its direct motion, gains a motion round the axis of the gun, by which the resistance on the preside of the bullet will be the same on all sides; or if it should be greater on one part than another; that part, by the circular motion, is presently transferred to the opposite side, and then it acts the contrary way; and such irregularities rectify one another; so that the ball will always go right forwards.

This may be explained by the motion of an arrow; for if an arrow that is not feathered, be shot from a bow, its motion will be very irregular; for if it be the least crooked imaginable, it will move towards that hand where the concave side lies. But when it is feathered truly, to give it a circular motion and make it spin, the concave part is turned every way, so that it will always fly streight forward. See Exam. 36th Fig. 220, of my Mechanics, 4to.

But in your common guns that are not rifled, I know no way to prevent that deflexion, but to polish the inside of the barrel, and oil the bullet when it is charged; for by this means the friction within the barrel will be made as small as possible; except you chuse to shoot with an oblong bullet as before-mentioned.

P R O P. XI.

Let d = diameter of an iron ball, $q = 7$, its density, $p = .00117$, the density of air, b = the velocity of the ball at first, v = velocity after it has described

Fig. *scribed the space x . $T = \frac{8dq}{3bp} = \frac{16000d}{b}$. Then*
 x be any space which the bullet describes after it
fired, $\frac{v}{b}$ is the number of the logarithm $\frac{-x}{36841d}$
all in feet.

This is demonstrated in Cor. 2. Prob. XIII. Sect. III. Fluxions. Supposing the resistance of the air to be as the square of the velocity, and supposing the elevation to be small, that the tract may not differ much from a right line.

Suppose the diameter d of the ball to be 6 inches then $d = \frac{1}{2}$, and $\frac{v}{b}$ is the number of the log. $\frac{-x}{18420}$

And suppose the space described x to be 200, 400, 600, 800, 1000, &c. feet, and b the first velocity to be 1. To find the velocity at the end of these several distances. The quantities reduced, we shall

have $\frac{-x}{18420} = -1 + \frac{18420 - x}{18420}$ for the logarithm, and the number of it $\frac{v}{b}$ or v .

x .	Logarithm.	Number v .
feet 200	— 1.98914	.9753
400	— 1.97829	.9512
600	— 1.96743	.9277
800	— 1.95657	.9048
1000	— 1.94571	.8825
1200	— 1.93485	.8607
1400	— 1.92400	.8395
1600	— 1.91314	.8187
1800	— 1.90228	.7985
2000	— 1.89143	.7788
2200	— 1.88056	.7596
2400	— 1.86970	.7408
&c.		

By this computation it appears, that if the spaces Fig. described be taken in arithmetical progression, increasing; the logarithms of the velocities will be arithmetical progression decreasing. For the differences of the numbers in the first col. x , are all equal; and so are the differences of the logarithms in the 2d column. And of consequence the numbers (v) in the third column, are in geometrical progression. And this is agreeable to what Sir J. Newton has laid down in Prop. V. Book II. Principia. Hence let the first velocity be what it will, an iron ball 6 inches diameter, after having moved 100 feet in the air, will proceed with $\frac{9}{10}$ its first velocity, having scarce lost $\frac{1}{10}$ of that velocity. And after moving through 2400 feet, will have lost little more than a quarter of its first velocity.

When $x = 18420$, then the log. is 1.00000, and the number of it $v = \frac{1}{10}$. So after it has moved 2400 feet, it still has $\frac{1}{10}$ its first velocity.

If x be greater than 18420, as suppose it be 20000, then the logarithm is $-1 - \frac{81580}{18420} = -1.44289$; that is -2.55711 , and the number $v = .03607$. So that except x be infinite, the velocity cannot be quite taken away.

Again,

Suppose the diameter of the ball 8 inches $= \frac{2}{3}$; then will $\frac{-x}{36841d} = \frac{-x}{24561} = -1 + \frac{24561 - x}{24561}$ be the logarithm of the velocity. If x be 1067, then the logarithm is -1.95657 , and the velocity $v = .9048$.

Again,

If the diameter of the ball be 2 inches $= \frac{1}{2}$, then $\frac{-x}{36841d} = \frac{-x}{6140} = -1 + \frac{6140 - x}{6140}$, for the

X

the

Fig. the logarithm; put $x = 267$; then the logarithm is 1.95656 ; and the number of it, or $v = .90$ for the velocity.

By all which it appears, that the same velocity remains in each ball, after describing spaces proportional to their diameters; as it ought to be by this law of resistance.

For the time.

The time may also be had from the same Co

For $v = \frac{Tb}{T+t}$, and $Tv + tv = Tb$, and $tv = Tb - Tv$, whence $t = \frac{b-v}{v} T$.

E X A M.

Let the diameter of the ball be 6 inches as before; then $T = \frac{16000d}{b} = 8000$. And let the velocity be $.8607$, then $b - v = 1 - v = .1393$ and $t = \frac{.1393}{.8607} \times 8000 = 1294'' = 21\frac{1}{2}$ minutes

for the time of flight, through 1200 feet. This is upon supposition, that the first velocity is 1000. But if the first velocity be 1000 = b , then $T = \frac{8000}{1000} = 8$, and then $t = \frac{.1393}{.8607} \times 8 = 1.294''$, the time of describing 1200 feet.

In this Prop. I have shewn the motion of an ball through the air, when its tract is not much different from a right line; and how much motion it has lost in moving through any given space, the air's resistance, and likewise the time of describing that space; all which is founded upon the theory of resistance laid down by Sir J. Newton. Therefore the resistance and retardation of a projectile cannot be much greater than is here de

ed; from whence it will follow, that consider-
the small retardation made in the motion of an
ball, the path described by a projectile cannot
very far from a parabola, especially in slow
ons.

Prob. XIX. Sect. III. *Fluxions*, I have calcu-
the tract or curve in which a projectile moves
resisting medium; by this Prob. its place may
ound in that tract, its velocity, and time of
ing, if any body will be at the pains to go
ugh the computation. But as it can be of no
er of use in gunnery, by reason of its diffi-
y, I shall say no more about it.

the line described by a ball in the air, is a curve
the hyperbolic kind, one of the asymptotes is
lled to the line of projection, and the other is
pendicular to the horizon. But the calculation
uch a curve is difficult, which makes it also use-
in Gunnery. And although by the resistance
the air, the curve described by a ball may differ
siderably from a parabola; yet since the com-
ations by it are so simple, and so easily done,
ought not to set the rules aside as useless, but
try to correct them from experiments, espe-
y as we have no other fixt rules to go by.
this end the following Prop. will be of use.

P R O P. XII.

*to construe tables for solving the common Problems
in gunnery.*

to do this, we must have a competent number
experiments, made with all sorts of guns at all
ations, to get their randoms, in manner fol-
ing. Take any gun, of which you know the
per charge, and elevate her to 5 degrees, and
her off, and measure the random, which set

X 2

down,

Fig. down. Then elevate her to 10, 15, 20, 25, &c. degrees, as far as 90, and find the random for each elevation as before; all which must be set down in a Table. But one shot for each cannot be relied on, but several must be made, in order to take a mean among them; so that the Table may be as perfect as possible.

This being made for one sort of guns, you must do the same for guns of all sizes, and make a Table for each, and then you will have a set of Tables, as complete as the nature of the thing will admit of. It is true this requires a great deal of labour and time.

When any Problem in Gunnery is proposed, and resolved by the rules derived from the parabola, and the solution compared with a similar one, which may be had from these Tables; the errors of these rules will appear, and afford means to the persons that are possessed of such Tables, of correcting the parabolic theory, which is the only one we have to calculate by. Therefore since we have several methods of computation, but what are founded on the parabola, and since they are so extremely near in themselves, and may be corrected so as to come near the truth. I shall here therefore solve several common Problems of Gunnery by the parabolic hypothesis, and shew how to proceed with the Tables for a solution, with which I shall conclude this Article.

P R O B. I.

The random or horizontal projection, at any elevation being given; to find the random at any other elevation, the charge remaining the same.

R u

R U L E.

The horizontal randoms, made with the same velocity, at different elevations; are as the sines of the doubled angles of elevation.

E X A M.

Suppose the random at 20 degrees elevation, be 58 feet, to find the random at 62 degrees.

As S. 40°	—	9.80806
to 3058	—	3.48543
S. S. 124 (or 56)		9.91857

13.40400

to 3944 (anf.)

3.59594

To solve it by the Tables of random. Seek among the Tables, till you find one that has the elevation and random, or the nearest thereto; in that Table against the other elevation, you have the random sought. This random you may compare with the other found by calculation, the difference will be the error.

Cor. 1. *The greatest random is at the elevation 45°.*

For the sine of 90 (2×45) is the greatest sine.

Cor. 2. *The greatest random is equal to the latus 55. sum of the parabola.*

For let $L =$ latus rectum, then since the angle $E = 45^\circ$, $AB = BE = 2BD$, but $L \times DB = BE^2$, that is $L \times \frac{1}{2}AB = AB^2$, or $L = 2AB = AC$. Or thus, since $DB = \frac{1}{4}AC$, B is the focus, and L the latus rectum.

S C H O L.

It is observed, that the greatest random is not at 45°, in the air, but only about $44\frac{1}{2}^\circ$.

X 3

P R O B.

G U N N E R Y.

P R O B. II.

The greatest random being given; to find the random at any given elevation.

E x A M.

Suppose the greatest random 5080 feet, to find the random at 25 degrees elevation.

As radius	—	10.
to 5080	—	3.70586
S. 50	— —	9.88425
Anf. 3892	—	<u>3.59011</u>

P R O B. III.

Having the random and elevation of the piece given to find the elevation proper for hitting a mark, at given distance.

E x A M. I.

Suppose the random 4900 feet, and elevation 42°; and the distance of the mark 3520 feet.

As 4900	—	3.69019
S. 84	— —	9.99761
fo 3520	—	3.51654
		<u>13.54415</u>
Anf. S. 45 36, or 134 25		<u>9.85396</u>

Half is 22 48, or 67 12, which are the two elevations, at either of which the mark will be hit.

E x A M. 2.

Suppose the greatest random 5080 feet, and mark 3520 feet distant,

ART. XIII. G U N N E R Y.

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The greatest random is at 45° , therefore

Fig.

as 5080 — 3.70856

to rad. — 10.

so 3520 — 3.54654

S. 43 52 or 136 8, 9.84068

half of which is 21 56, or 68 4; so that the mark may be hit, either at the elevation of 21 55, or the elevation 64 4.

By the Tables; find the given distance and elevation in some Table, in which, against the given distance, is the elevation.

Cor. Hence the randoms are equal, when the elevations are equally distant above and below 45° , being complements to one another.

S C H O L.

It has been observed, that in the air, the lower randoms go a little farther than the correspondent upper randoms. Also when the greatest random is taken from experiments, the randoms at lesser elevations, being computed by the lines of the doubled arches, always come out less, than the randoms obtained from experiments.

P R O B. IV:

Having the greatest random upon a horizontal plane; find the greatest random on an inclined plane.

R U L E.

The projections or randoms made on different planes, with the same velocity, are in the compound ratio of the sines of the angles which the lines of direction make with the plane and zenith; and the squares of the cosines of the plane's elevations reciprocally.

X 4

And

Fig. And the greatest randoms on different planes are as the versed sines of the angles between the plane and zenith directly, and the versed sines of double these angles reciprocally; either for ascents or descents.

Or rather thus; the randoms are reciprocally as the versed sines of the supplements of the angle between the plane and zenith.

E X A M.

Suppose the greatest horizontal random 3990 what is the greatest random on a plane elevated or depressed 9 degrees.

For the ascent.

Here $\frac{v.90}{v.180} = \frac{1}{2}$, and therefore $\frac{1}{2} : \frac{v.81}{v.162} :: 3990 : \text{random sought} = \frac{2 \times v.81^\circ \times 1990}{v.162}$, for the ascent.

Log. 2.	—	0.30103
vers. 81 (.84357)	—	1.92611
3990	—	3.60097
		3.82811
vers. 162 (1.95105)		0.29027
Ans. 3450	—	3.53784

For the descent.

The random sought = $\frac{2 \times v.99^\circ \times 3990}{v.198 \text{ or } 162}$

Log. 2.	—	0.30103
ver. 99 (1.15643)		0.06311
3990	—	3.60097
		3.96511
vers. 198 (1.951)		0.29027
Ans. 4730	—	3.67484

Or thus.

Rad.	—	10.
399°	—	<u>3.60097</u>
		13.60097
vers. 99°	—	<u>10.06312</u>
345°	—	<u>3.53785</u>
		13.60097
vers. 81	—	<u>9.92612</u>
473°	—	<u>3.67485</u>

Cor. The greatest random on an inclined plain, is when the line of direction bisects the angle between the zenith and the plain. And the randoms are equal which are made at equal distances above and below this line.

P R O B. V.

Having the greatest horizontal random, to find the elevation to hit an object with the least force, whose height and distance are given.

The angle of elevation is found thus; say as the distance, to radius; so the altitude of the object, to the tangent of its elevation. But this elevation may be more easily had by observation. Then suppose a line drawn from the place to the object; then the line of direction sought must bisect the angle between this line or the object, and the zenith. And this will be so whether the object be above or below the plain of the horizon.

Cor. To hit an object with the least force, the line of direction must bisect the angle between the zenith and the object.

P R O B.

G U N N E R Y.

P R O B. VI.

The velocity in feet, and the elevation being given to find the horizontal random.

R U L E.

Let $f = 16.1$ feet, $A =$ sine of double the angle of elevation, $v =$ the velocity. Then $\frac{vvA}{2f \times \text{rad}}$ = horizontal random.

E X A M.

Let the elevation be 30 degrees, and the velocity 350.

350	—	2.54406
350	—	2.54406
S. 60	—	9.93753
		15.02565
32.2	—	1.50785
rad.	—	10.
		11.50785
Anf. 3295	—	3.51780

Cor. 1. *The elevations being the same, the horizontal randoms are as the squares of the velocities.*

56. Cor. 2. *The velocity is a mean proportional between 16.1 and the parameter to AF, at the point of projection.*

For if AG be the velocity, or space moved in second; then $GH = 16.1$, the space descended in a second. But $\frac{AG^2}{GH} = \text{parameter to AF} = \frac{vv}{f}$.

Cor

Cor. 3. *The horizontal velocity is constantly AI. Fig. And the principal latus rectum of the parabola = 56.*

$$\frac{AI^2}{f}$$

Cor. 4. *The velocity at any point A of the same parabola, is as the secant of the angle IAG, or reciprocally as the cosine of the elevation.*

For AI being constant, $AI : AG :: \text{rad.} : \text{secant of IAG} :: \text{cos. IAG} : \text{to radius.}$

Cor. 5. *If DE be made equal to $\frac{1}{4}$ the latus rectum, 57. and the horizontal line GEH drawn; the velocity at any point F in the curve, is that which a body acquires by falling from the line GF to that point F. Thus the velocity at D, is that by falling through ED, and at F through GF, &c.*

For if $DE = DO = \frac{1}{4}$ the latus rectum, then O is the focus, and OK or $DI = 2DE = 2IK$. The velocity in falling through ED is the same as in falling through DO; but in an equal time it will describe $2DO$ or DI , or OK , and therefore will pass through the point K of the parabola. And therefore the body will describe the parabola whose focus is O, and $\frac{1}{4}$ the latus rectum ED; and consequently the velocity at F is that which is acquired by falling through GF. But (by the Conic Sections) GF is $\frac{1}{4}$ the latus rectum to the diameter FN.

Cor. 6. *A body projected perpendicularly upwards, with the velocity at F in the parabola DFA, will ascend to the height FG or PE.*

P R O B. VII.

Having the random made with a certain charge of powder, at a given elevation; to find the charge of powder, for any other random, with the same elevation.

R U L E.

G U N N E R Y.

R U L E.

The charges of powder will be nearly as the randoms, excepting that the charge is something too great, when it is less than the given charge; and too little, when it is bigger. See Cor. Prop. II. But if Prop. III. be admitted, the random will be exactly as the charge.

E X A M.

If 8lb. of powder throw a ball 4010 feet; how much powder will throw it 5700 feet?

$$4010f : 8^{lb.} :: 5700f$$

8

$$\begin{array}{r} 4010 \overline{) 45600} (11.3^{lb.} \text{ answer.} \\ 4010 \cdot \\ \hline 5500 \end{array}$$

$$\begin{array}{r} 5500 \text{ (but something} \\ 4010 \text{ too little by Prop. II.)} \\ \hline 1490 \end{array}$$

$$\begin{array}{r} 1490 \\ \hline \end{array}$$

P R O B. VIII.

The height of the perpendicular projection and elevation, being given; to find the height of the shot.

R U L E.

The altitudes of projections are as the versed sines of the doubled angles of elevation, when the velocities are given; and as the squares of the velocities, when the elevation is given.

E X A M.

Let the height of the perpendicular projection be 3420 feet, what is the height when the elevation is 25 degrees?

As

ART. XIII. G U N N E R Y.

317

As vers. 180 — 10.30103

3420 — — 3.53402

vers. 70 — — 9.81821

13.35223

alt. 1125 — 3.05120

3.05120

Fig.

57.

Cor. 1. If P = perpendicular projection in feet,
then $v = 2\sqrt{Pf}$ the velocity.

For $\sqrt{f}$ (height) : $2f$ (velocity) :: $\sqrt{P}$: v .

Cor. 2. If B = vers. of twice the elevation,

$\frac{PB}{8f} = \frac{PB}{2 \text{ rad.}}$ = the altitude.

Cor. 3. The perpendicular projection is the greatest altitude.

For the vers. of 180 = 2 radius.

P R O B: IX.

Given the height of the perpendicular projection, and the elevation; to find the random.

R U L E.

As radius, to the perpendicular projection; so the sine of twice the elevation, to half the random.

For the random = $\frac{2AP}{r}$.

E x A M.

Let the perpendicular projection be 3420, and the elevation 35°.

Rad. — 10.

3420 — 3.53402

S. 70 — 9.97298

½ random, 3214, 3.50700random 6428

Cor:

Fig. Cor. *The perpendicular projection is half the greatest random.*

S C H O L.

It has been observed, that from the resistance of the air, the perpendicular projection is something more than half the greatest random.

P R O B. X.

The random and elevation given; to find the height of the shot.

R U L E.

As radius, to the tangent of elevation :: $\frac{1}{4}$ the random, to the height of the projection.

56. For $AB : BE$ or $2BD :: \text{rad.} : \tan.$ $BAE :: \frac{1}{2}AB : BD.$

E X A M.

Let the random be 6428, and elevation 35.

Rad.	—	10.
$\tan.$ 35	—	9.84522
$\frac{1}{4} \tan.$ 1607	—	3.20601
height, 1125,	—	3.05123

P R O B. XI.

The random and height of the shot being given; to find the perpendicular projection.

R U L E.

Let $D = \text{random}$, and $d = \frac{1}{4}D$, $b = \text{height of the projection}$; then $b + \frac{dd}{b} = \text{height of the perpendicular projection.}$

This appears by Cor. 6. Prob. VI.

E X A M.

E X A M.

Suppose the random be 6428, and height 1125.

Then $d =$ 1607

1607

11249

9642

1607

1125)2582449(2295

... 2250 .. 1125

3324

2250

3420

anf.

107

101

6

Cor. 1. *The principal latus rectum of the parabola*

DD

$\frac{4b}{a}$

Cor. 2. *The velocity of projection is $2\sqrt{fb}$.*

P R O B. XII.

Given the elevation, and random; to find the time of flight.

R U L E.

The times of flight are as the sines of the angles of elevation.

Or thus, as rad. : tan. elevation :: random : Q.

Then $\sqrt{\frac{Q}{f}} = \text{time in seconds.}$

E X A M.

E X A M.

Let the elevation be 35, and random 6428.

Rad.	—	—	10.
tan. 35	—	—	9.84522
6428	—	—	3.80807
Q	—	—	3.65329
f = 16.1	—	—	1.20682
			2.44647
Ans. 16.7"	—	—	1.22323

Cor. The height of the shot is $\frac{1}{4}Q$, being the height of the parabola.

S C H O L.

58. The rules here laid down are mostly derived from the motion of projectiles in my *Mechanics*. Besides the arithmetical calculations, the several Problems may be solved geometrically, after the following manner. Let AB be the perpendicular projection of the shot. About AB describe the semicircle ACB. Make AK = 2AB, which will be the greatest random. Let AH be any other random, and make AE = $\frac{1}{4}AH$, and erect EC perpendicular to AH to cut the semicircle in C, c; and draw ACD, Acd. Make AF = $\frac{1}{4}AH$, and raise Fd perpendicular to AH. Bisect FD, Fd in G and g, which will be the vertexes of two parabolas passing through A, H, and touching the lines AD, Ad, and whose parameters are 4BL, 4Bl. Therefore if EAC, EAc be two elevations, so that CR = cR, the projections made in these two directions will describe these parabolas, and both pass through H. And if AI = $\frac{1}{4}AK$, and IR touch the circle in R, AR will be 90°, and the projection made along AR will pass through the point K. So then AGH, AgH will be the paths of the projectiles, in direction AC

Ac, and whose velocity is that acquired by fall-
g through BA. And the angle ACB or AcB is
ways a right angle. And the heights of the pa-
rolas will be $FG = EC$ or AL , and $Fg = Ec$ or
being the versed sines of twice the elevation.
And the time of ascent and descent in AB, to the
in the parabola AGH (or AgH), is as AB to
C (or Ac). Also LC (*lc*) is the sine of twice the
elevation = $\frac{1}{4}$ the random.

When any place is to be bombarded, it may be
either by elevating the piece to a certain an-
gle, and increasing or diminishing the quantity of
powder, till the ball be just thrown to the mark.
Or else by taking a proper charge of powder, and
elevating or depressing the piece till she just shoots
to the place; and then keep to that charge and ele-
vation. If the gun is elevated to 45 degrees, the
best charge of powder will then do the business;
and by this, much powder will be saved. In bom-
barding houses, it is best to elevate the gun above
45°, for then the bomb rises to a greater height,
and so will do more mischief. But where the bomb
is in the streets, or is shot among a parcel of
houses, it is better to elevate below 45°, for then it
will not bury itself in the ground. And at sea,
mortars should be elevated above 45°, for the agi-
tation of the sea continually changes the direction
of the mortar, and would otherwise render the shot
very uncertain; but in great elevations, several de-
grees alteration make very little difference in the
effect.

Y

A R T

Fig.

A R T. XIV.
 A R C H I T E C T U R E
 O R T H E
 A R T O F B U I L D I N G.

THIS art teaches how to erect any kind of buildings, as palaces, churches, or private houses. In performing of which, many hands are employed of various branches of business, as masons, stone-cutters, bricklayers, carpenters, joiners, smiths, plumbers, glaziers, tylers, slaters, &c. This is an art of such extensive use and benefit that there is no such thing as mankind living without it; the inhabitants of all countries have contrived methods to build themselves houses to live in; by which they are preserved from cold, snow, frost, rain, and all inclemencies of weather; and in which they can preserve from destruction, their victuals, cloaths, bedding, and all other necessaries they make use of. This art then must be ranked in the first place, for till that has been put in execution to procure a dwelling for any person, he can practise no other art.

In every building three things must be principally considered, *strength*, *conveniency*, and *beauty*. *Strength* requires good materials and good workmanship. *Convenience* requires, that the several parts be so ordered and disposed that one part may not be a hindrance to another; that those parts which one has most recourse to, should be the nearest at hand; that the room or bigness of every part be according to the use of it, or the business to be done in it.

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Beauty arises from the form of it, and the correspondence of all the parts with the whole, so as to be agreeable to nature.

When a building is to be erected, the expence of it ought to be carefully calculated, by some judicious person, and timely provision made for money, that the workmen may not stay for their wages beyond the due time. And a sufficient stock of materials must be provided and got in, that there be no stop in going forward with the building, for it is an advantage to the work to be carried on and finished with expedition. The materials to be taken care for are timber, stone, bricks, lime, sand, and iron. As to the choice or goodness of these, and the manner of using them, I shall explain in the following Propositions.

P R O P. I.

To describe the several materials, as timber, stone, sand, &c.

The sorts of timber mostly used in building are oak, elm, ash, and fir. The oak will warp and twist, except it be many years old, and thoroughly dry; but the elm will cast more than any other; the ash not so much. But fir is the best wood for standing its life, for it is of a springy nature, and will cast very little. All sappy parts must be cut out of wood used in building, for it soon decays. The most proper time of felling timber is the winter season, and some say in the decrease of the moon. In the winter the sap is all expended in feeding the branches, and wood felled then will neither warp nor cleave; but felled in summer it will rift and shrink. It is therefore a wrong method of felling oak in the summer, for the sake of the bark; for the wood will not be so strong nor so

Fig. good. A better way is to bark the trees in the summer standing, and let them stand till the winter. For after the bark is taken off, the trees will live and thrive till winter. The timber of such trees as grow in open fields, is better than such as grow in close woods and shady places. Some say, that the faster the tree grows, the stronger the timber will be; and that good ground produces better timber than bad ground. The age of a tree may be known, when it is sawn through, by the number of rings round the pith; in each ring there is a number of open pipes, which convey the sap upwards. As many of these rings, so many years old the tree is. If you observe whereabouts the rings are widest; that side of the tree faces the south. No timber should be used in building till it be three years old. Fir wood is the best for roofs and joists, and for flooring and doors. But oak is the best for window-frames or door-posts, or any part that faces the weather, for it will last beyond any other sort of wood.

Stone is of various kinds, as marble, alabaster, fire stone, purbeck stone, free stone, common quarry stones, pebbles, or river stones, Portland stone, &c. There are several sorts of *marble*, as white, black, grey, green, veined or spotted. It is a very hard stone, and capable of a fine polish. It is used for chimney-pieces, hearths, tables, tomb stones, and statues. *Alabaster* is a soft, white sort of marble; it is used for monuments and carved figures, or to make cement. *Fire-stone*, called also rigate stone, is used for hearths, ovens, stoves, &c. *Purbeck stone* is a hard grey stone, used for pavements, they may be cut square and polished. *Free stone*, this is dug out of quarries, being a soft whitish stone, and of a fine sandy greet; and may be squared and wrought into any form. It is used for flags, jaumbs for windows and doors, and in building

building the fronts of houses, and the arches of Fig. bridges, &c.

Quarry stones, are dug out of a rock or quarry; they will not work to any form, being very hard. They are very good for building common walls. *Stubbles* are such stones as lie loose in the bottom of a river, of all forms, and very hard. They are used for common pavements in the streets, and for coarse walls. *Portland stone*, is brought from Dorsetshire, it is softer and whiter than Purbeck, and is used in greater blocks out of the quarry. This is much used in building, but it will not endure fire.

Lime, this is made of burnt stones, and these stones are either dug out of quarries, or gathered out of rivers. The quarries proper for this are called lime-stone quarries. That lime is best which is made of the hardest stone, and when burnt is of a blueish colour. Quarry stones are better than those gathered stones. Lime may be made of shells, as cockles, oysters, &c. And about London they frequently burn chalk for lime; but it makes very weak lime; and is fittest for plaistering, being very white. When the lime stones are well burnt, which requires two days or more, it is taken out of the kiln, and led away. Then it is to be *slaked*, which is done by turning it over with a shovel, and throwing as much water upon it, little by little, as will make it fall to powder. When it is all thus slaked, it must be lightly covered with a lid. It ought to lie thus for a week at least, before it is made up; but if it lie longer it is better; for it requires time for all the knots to dissolve. When it is to be made into mortar, it must be mixed with sand; if it be river sand, you must put two loads of it to one load of lime, but pit sand being softer, take three loads of it to a load of lime. The lime and sand must be well mixed together,

Fig. and made up with water. But it is not to be used directly, but must lie some time.

Sand, this is had several ways; by digging in the ground, or leading it out of the rivers, or from the sea. The river sand makes the strongest lime but less of it is used. Sea sand is reckoned the worst, by reason the salt makes it moulder away therefore not so fit to make lime with. Sand is best when dirt is mixed with it.

Metals, which are made use of in building, are iron, lead, and copper. *Iron* serves to make nails, crooks, bars, hinges, grates, &c. English iron is the worst for making nails of, &c. for it is so brittle the nails will not clinch. Iron is good, if it appears of a fine grey colour when broken; but bad when many glittering specks appear in it. *Lead* serves for glazing with, and sheet-lead serves for covering the roofs of churches, and any state buildings, and for lead gutters, and pipes for carrying off the water. And is serviceable for fastening any iron work in stone. *Copper* is used for making nails, clasps, &c. Also *brass* is used much in the same way.

Bricks are a sort of artificial stone. They are made of clay, dug and seasoned in the winter, and burnt in the summer. There are several sorts of bricks, as *common bricks*, from 9 to 11 inches long. *Hollow bricks*, flat on one side and hollow on the other, when the hollows are laid together, they make conduits for conveying water away. *Flemish bricks*, a small sort for paving with. *Feather-edge bricks*, are thinner on one side than the other. *Paving bricks*, or *paving tiles*, of different forms and sizes, for laying floors with. *Place bricks* and *stock bricks*, for walling. *Statute bricks*, such as are made about London, according to act of parliament, they are to be 9 inches long. *Copeing bricks* for the tops of walls. And several other sorts.

PROP. II.

To lay the foundation of any building.

The greatest care ought to be taken to lay a good foundation, for if that fails, the super-structure cannot be supported. And generally speaking, there are more errors committed this way, than in any thing else, and are of the most pernicious consequence, being often attended with the ruin of the whole building. Foundations are to be laid within the ground, and so much the deeper as the earth is softer and looser; and you must dig till you come at firm ground. The best foundation is when we build upon a rock, the next is a hard and firm gravel; a strong stiff clay is also very good; but soft loose earth is bad; and the worst is poorish soft earth; and among quick-springs. Good ground may be known by striking forcibly with a heavy body, and then if it be firm it will not shake or sound, or move the water in a vessel. The foundation must be digged wider than the thickness of the wall, to make offsets, and so much wider as the ground is looser; they must be very level at the bottom; and in laying your stones, place the flattest side downwards upon the level earth, or if you set an angle or edge of it downwards, the weight will make it sink into the earth like a wedge, and make the building shrink. You must make offsets at the bottom; sometimes two or three must be made. But never lay a stone further out for an offset than a third part of its length. The looser the ground is, the more offsets must be made; for observe, the greater the base on which your foundation stands, and the more points the foundation stones touch in, the firmer it is, and the more weight it will carry. The offsets must all

Fig. be lower than the floor within. ABCDE is a wall with its offsets. Some workmen, after the foundation is laid, begin directly to build upon it, without filling up the hollows or vacancies at the bottom; but this is wrong; for when the outside stones are laid, the inside must also be filled with stones truly laid; and when it will hold no more earth must be rammed in between them, and all under such stones as will not lie level, and the earth well beat. If this is not done, it cannot be called a good foundation.

In soft boggy ground which sinks much, piles must be driven into all the foundations, as thick as they can stand, and then planks laid over them and pinned fast, upon these you begin to build, and this is absolutely necessary in laying the foundations of the piers of bridges, where the ground is not found. And the piles must be driven to the sound ground.

In a large house the foundations must be twice as thick as the walls; and as much offset made on one side as on the other, or rather something more on the outside.

If the ground is bad only in some particular places, and the rest good; arches may be turned over the bad places, which will take off the weight

PROP. III.

To describe the method of walling.

The materials for walling with are stone and brick, and mortar made of lime; and in walling with either, observe these particular directions.

1. The walls of any house must be exactly perpendicular.
2. The strongest and greatest materials must be at the bottom.
3. The walls of a house must diminish proportionally in thickness, even to the top.

story in height. 4. That for every foot and half, Fig. 59. or two feet high, a course of *throughs* be laid; these throughs go quite through the wall, and serve to bind it together, which otherwise would split. The corners must be well tied together, with *quoins* laid cross and cross, of a good length, these quoins are commonly squared stone. 6. No wall should be wrought above three feet high before the next adjoining walls be brought up, and these in their turn must be carried above the rest, but not more than three feet. If the walls are not thus bound into one another, they will crack and part. In walling, observe to break the joints, that is, not to make one joint above another, which is bad workmanship, and may cause the walls to crack; or the stones or bricks ought all to be tied over one another as much as they can. 9. Avoid making great joints, but lay the bricks or stones as close as they can be put, with narrow joints of lime. 10. In all buildings, the solid parts must be directly over the solid parts, and the vacant parts over the vacant. Thus pillars must be above pillars, windows and doors above windows and doors, and walls above walls, partitions above partitions; to make good work.

Brick walls are built, by laying the bricks lengthways upon the wall, end for end, in a due quantity of lime; this is called a *stretching course*. And when the wall is raised about every six courses, then a course of bricks is laid with their lengths cross the wall, called a *heading course*. This binds the wall together. But when a brick does not reach through the wall, a stretching course more is laid on one side of the wall than on the other. And when two bricks laid cross must tie over one another, for all the length of the wall. Sometimes the wall is so thick, that 3 or 4 bricks must be thus tied together cross. Stone walls are built much

Fig. much after the same manner, if they are made
 59. hewn stones. Every wall consists of two walls
 two parts, an outside and an inside wall; then the
 middle part is filled with lime and rubbish stones.
 Walls built of pebbles taken out of rivers, are
 built by no certain rule; for in both sides of the
 wall, the stones must be laid as they will fit the
 places; and the inside is filled with mortar and
 rubbish, called filling stones. And at every half
 yard high the wall is levelled, and a course of through
 is laid on, as I said before. The form of a brick
 60. wall, see Fig. 60.

When cellars are to be made; their foundation
 must be laid 7 feet below the first floor; for the top
 or ceiling must lie level with the ground on which
 the house stands, or very little higher. They ought
 to be entirely within ground, and to be situated on
 the north side of a house, for coolness. They are
 best arched over-head; for if they be covered above
 with a wooden floor, the wood soon decays, by
 the great dampness.

P R O P. IV.

To shew the method of laying different sorts of floors

Ground floors are made various ways, as with
 bricks, flags, or marble. Square bricks from 6
 to 12 inches, are made on purpose for flooring
 and make a very good floor. Common bricks are
 also used for floors; they are laid length-ways cross
 the house by a line, one course after another, but
 so that the joints at the ends may fall about the
 middle of each brick in the former course; this
 way they are laid flat, but they may be set edge-
 ways, and then they make a very strong floor, but
 consume twice as many bricks. Before they begin
 to lay them, they level the floor very exactly, and then

then lay a layer of sand upon it, and spread it level, three or four inches thick. Flags are laid after the same manner; they must first be dressed plain on the upper side, and then made square on all sides, and they must be all made of the same breadth; and then they are laid course after course by a line, cross the house; so that there may not be a joint upon joint.

Very good floors are laid with lime and gravel after this manner. After the lime has been slaked, and has laid some time, it is riddled through a wire riddle, to take all the lumps out of it. Then the sharpest gravel that can be had must be got, and all the roughest part taken out by riddling; what remains consists of very coarse sand, mixed with very small stones. Put an equal quantity of this gravel, and lime together; and make it up like mortar. Then the house floor is to be leveled, and spread over with this mortar to the thickness of about 3 or 4 inches, and made very level. It must be left thus for a few days till it hardens; and when a person can walk on it without sinking, he must take a beater made of a piece of flat board, and beat it all over very level; and when it begins to crack it must be beaten again. And thus it must be beaten six or eight times over, at the distance of 4 or 5 days, or a week, till it grows hard, and crack no more. But care must be taken that nothing comes upon it, till it be thoroughly dry. Instead of gravel some mix coals with the lime, and others smiddy gum (being the flakes that fly off hot iron) when it can be had. But the best thing is *sparr* beat small, which comes out of the lead mines.

Flooring with marble is the most beautiful as well as the most costly. It is laid in one entire stone before chimnies. And it is often laid of two colours placed chequerwise or square, side by side; or

Fig. or arrafs wise, laid angle to angle. And there are several forms contrived to lay pieces of different colours among one another, which appear very beautiful.

The cheapest flooring is that made with common rough stones, either gathered out of rivers, or raised in common quarries. This is proper for streets, courts, common high-ways, &c.

Floors of upper rooms are laid with deal boards or fir boards. In building a house, joists must be laid along to nail the boards to. These joists must be made deeper than broad, for the greatest part of the strength lies in the depth, for the strength increases as the square of the depth, and only simplifies as the breadth. The depth may be 6 or 8 inches or more if they are long, and be to carry a great weight. And they may be laid a foot or half a yard from one another, but not more than 2 feet. The boards must be plained and set to dry. These boards ought to be 2 or 3 years old, or else they will pine, when laid in a floor; for they ought to be very dry when laid. And the broadest board will pine the most; therefore to make good floors so that the joints may not appear, they use narrow boards, called *battens*. These boards when straight are channelled on the edges, and lapped and driven close together, and nailed down with brads or unheaded nails, which will sink into the wood; and then the floor is plained smooth and even.

No dormants must be laid over the heads of doors or windows.

P R O P. V.

To shew the proportion of the several parts of a room to one another.

An entry or passage ought to be so situated, as to have communication with all the other parts of the house.

use; it is a common place for people to stay in, Fig.
 who are waiting to negotiate any business. This is
 the first part of the house that one enters into. It
 serves for a passage to and fro betwixt other rooms,
 and from the out-door into the house. Its dimen-
 sions depend upon the other parts of the house.

The *hall*, this is a large room, which serves for
 feasts, acting plays, or any grand entertainments.
 These places ought to be very large, to entertain
 many persons. The length of a hall ought at least
 to be twice the breadth; or in large buildings,
 three times the breadth. The height thereof ought to
 be $\frac{1}{2}$ the breadth, or it may be 16 or 18 feet in
 great buildings. They are sometimes arched over,
 and then the height from the ground to the top of
 the inside of the arch must be $\frac{2}{3}$ the breadth. Like-
 wise the other low rooms may be arched.

Anti-chamber, is an outer-chamber in great houses,
 where any persons are to wait that go upon busi-
 ness. The breadth to the length should be as 5 to
 7 or a little more; and the height $\frac{2}{3}$ of the breadth,
 some say $\frac{3}{4}$ the breadth. But if it be arched,
 the height must be $\frac{2}{3}$ the breadth.

Anti-chambers and chambers ought to be so si-
 tuated, that they may be on each side of the entry
 hall; and nearly to answer alike on both sides.

Of *chambers*, all the rooms above the first story
 are called chambers, only those at the top of the
 house are called *garrets*. The length of a cham-
 ber ought to be once and a half the breadth, at
 the most. And the height must be $\frac{3}{4}$ the breadth.
 Some say the length must be $1\frac{1}{3}$, or $1\frac{1}{6}$, or $1\frac{1}{7}$, or
 the breadth; so this is all arbitrary. But if the
 chambers are arched, the height must be $\frac{2}{3}$ the
 breadth, measuring from the ground to the top of
 the arch within.

Chambers of the second story must be a twelfth
 part less in height than the chambers below. But
 if

Fig if they be arched, they must be a sixth part less than those below. In chambers of the third story the height must be $\frac{3}{4}$ the height of the second chamber, or that below it. The dimensions of garrets are left to the judgment and fancy of the builder; and must be contrived according to the particular occasions and uses they are designed for.

In the building of chambers, regard must be had to the situation of the chimney, the place of the bed, the places of the windows; so as to give them all such situations as may best suit the purpose. The bigness and place of the passage must also be considered.

Galleries are a sort of chambers made out of the house, commonly in the front; and are built upon pillars. They are long narrow rooms, made for walking, eating, and taking diversion in. Their length must at least be 5 times their breadth, but not more than 8 times. The lower galleries must be as high as the chambers of the first story; so that one may walk on even ground out of the chamber into the gallery. Sometimes galleries are made so that one may go into them out of the court without coming into the house; and then stairs are made to ascend into them. And in this case the galleries may be lower than the first story. Upper galleries must be as high as they are broad, if they be flat; but if arched, they must be a third, fourth or fifth part more.

The low rooms, as the entry, hall, &c. are all upon a level, but are raised 2 or 3 feet higher than the court before the house; from which one ascends into the entry by steps which may be 4, 5, or 6 inches high, but not more; and a foot or 12 inches in breadth.

PROP. VI.

To describe the bigness of doors and windows, and the proportion of the parts.

The out doors of a large house must be proportioned to the bigness of the house. Let the doors and windows in any house be as few in number, and of no greater dimensions than is necessary for free passage, and giving a due quantity of light. For all openings are so many weakenings. And let them not come too near the corners of the house, for then the corners will be weakened, which is the greatest strength of the house. Let the doors be over doors, and windows over windows, for it makes the work very weak to have a wall over a void place. Some say the doors ought to be placed so, that one may see through them all from one end of a house to the other, but this cannot always be done without losing greater conveniences. It makes strong work to turn arches over the doors and windows, which will discharge a great deal of weight. Inner doors must be six feet high, and two feet and a half, or two feet nine inches broad. If doors be three feet wide, or more, they must be twice as high as broad.

Windows must be made so as to give a due quantity of light, for a room may have too much light as well as too little. A great room requires more light than a little room. And regard must be had to the situation of the place; if it be shaded, or towards the north, it requires the more light. In large houses, the windows must be equal to one another in the same story, and must stand in the same rank and order; and so that those on the right may answer those on the left; and likewise those above to those below.

In

Fig. In small houses the breadth of the windows within, may be $2\frac{1}{2}$ feet, or 3 feet, and the height $3\frac{1}{2}$ feet or 4 feet. But in large houses, they may be 4 or 5 feet within the jaumbs, or in greater buildings 6 or 7 feet. And the height double the breadth and sometimes a third or fourth part more.

Windows of the second story must have the same breadth, but they must be lower by a twelfth part; but some say they must be lower by a third part. And the windows in the third story must be a fourth part lower than those in the second.

The height of the window soles must be 2 feet 8 inches, or not above 3 feet. In small houses the window frames must be $2\frac{1}{2}$ or 3 inches thick, which must be of oak. But in large houses they are made 4 or 5 inches thick. When a window is wide, each shutter must open in the middle with joints. Door-posts of oak ought to be 6 or 8 inches square in large houses, and 4 or 5 inches in small houses.

Pilasters of doors and windows must be $\frac{1}{4}$ or $\frac{1}{5}$ part of the aperture, and must project $\frac{1}{6}$ part of the thickness.

There are several ornaments for doors and windows, as the architrave, frize, and cornice. The architrave ought to be as thick as the pilaster; and the frize above that must be $\frac{3}{4}$ the height of the architrave, and the cornice at the top must be $\frac{1}{4}$ the height of the architrave. Their ornaments and mouldings are made according to that order of columns, which the builder thinks proper.

P R O P. VII.

To give an account of the construction of chimnies and the methods attempted to cure their smoaking.

In ordinary houses the breadth of a chimney may be two feet, the height 3 feet 8 inches, or 2 feet

at 2 inches above the bars; and the depth back-wards 15 inches within. The top or mantle-tree to project 5 inches over the jaumbs, and the bars set 10 inches further out than the jaumbs: these are convenient measures. But in large buildings, hall chimnies must be 6 or 7 feet wide, and in great buildings 8 feet between the jaumbs; their height 4 or 5 feet, and depth backward 2 or 3 feet. Chamber chimnies must be 5, 6, or 7 feet wide, and 4 or $4\frac{1}{2}$ feet high, and 2 or $2\frac{1}{2}$ feet deep. Lesser rooms may have the chimnies 4 or 5 feet broad, 4 feet high, and 2 feet deep.

If it can be done, make the funnels in the thickness of the walls, which will save a deal of room in the house; and these funnels must be carried through the roof 3, 4, or 5 feet, to convey the smoak away. They must be made of a proper wideness; for if they are too wide, the wind will drive the smoak back into the room; and if they be too narrow, the smoak cannot have free passage. No chimney-shaft should be straiter than 10 inches; chamber chimnies may be 10 inches deep and 15 wide; and hall and kitchen chimnies 15 inches both ways. The shafts must be carried up very streight. It often happens, that making the chimney-piece too high causes the room to smoak; and therefore it is better to make them lower than these stated rules.

There are several causes for the smoaking of chimnies. If a house is so situated, that it be overtopped by other houses, or by trees, hills, &c. they are apt to smoak, for the goal that comes behind these obstacles, will strike down the chimney; except these obstacles be very high above your chimney, for then the wind flies quite over, and does no damage. But if it happens that the wind comes contrary, and blow against another house higher than yours, and beyond yours, and very near it,

Fig. it is hardly possible yours should avoid smoaking. But there may be some fault in the building, by making the funnel too wide or too narrow.

Several ways have been tried to cure smoaking chimnies, which sometimes succeed. One way is to observe what wind makes the house smoak; and on that side, make a hole in the top of the chimney about the middle, in which fix a flat slate sloping upwards. The wind blowing in there will go out of the chimney top, and the smoak will follow.

Another way is to place an eolipile in the chimney, where it can be heated, and the vapour going out of it will drive the smoak up the chimney.

Sometimes a vane is put at the top of the chimney, which carries about a broad piece of tin lying asslant from the wind. This tin being always turned against the wind, shelters the smoak, which goes out freely.

Covering the top of the chimney with a broad stone, set 5 or 6 inches above it, being supported at the corners by four pillars, will sometimes prevent smoaking, when the wind tends directly down the chimney. And sometimes covering the top with two slates lying aslope like the roof of a house, leaving two opposite sides open, will cure it.

Sometimes the doors of a room are badly situated, and the shifting of them has cured the smoaking.

When the fire-place is too wide, the chimney apt to smoak; and then it is prevented by building two cheeks within it, to make the passage narrower. For the straiter the passage, the swifter the fresh air will rush in towards the fire, which being rarified, will naturally ascend up the chimney, and carry the smoak with it.

When the funnel of a chimney is made too narrow, there is no remedy for the smoaking, but to keep a very small fire.

Wh

When the chimney is in an out-wall; making a Fig. 1. in the fire-place, through the wall, to let a fresh supply of air come in, often cures it. For the closeness of a room is often the occasion of smoaking, for if a fresh supply of air does not come in, to make up for what is expended in carrying out the smoak, one may be sure the smoak cannot pass off, but will fill the house.

Sometimes the smoaking may be cured by making two opposite sides of the chimney top, higher than the other two, by a course or two of bricks; which is to be done according to the position of the wind. For in almost all cases the wind has a hand in the smoaking of chimnies. A funnel put behind the fire with a mouth a foot square, sometimes cures the smoaking.

P R O P. VIII.

To describe the different methods of making stairs and stair-cases.

A stair-case ought to be so placed, that when the stairs are made, the communication with all the chambers may be easy and convenient; and the room in it ought to be proportioned to the rest of the house; it must have a due quantity of light, therefore if it have but one window, it must be placed in the middle, to enlighten the whole. There ought to be at least one half space before the stairs come at the landing-place on the top. For where there is but one flight of stairs, that is, where they go in one direction to the top, without any turning-places, it is very dangerous going down. For if one happens to slip or fall, they must go to the bottom at once, there being no turn, or any place to stop one. Yet such stairs are forced to be made sometimes into garrets where room is scarce.

Fig. The landing-place must be large enough to lead to all the chambers. In small houses the height of each step may be 8 inches, or not above 9; and the breadth a foot, or not less than 9 inches, and the length three feet. But in large houses the height of a step must not be more than 6 inches, nor less than 4, their breadth not less than 12 inches, nor more than 16. And the length 4, 5, or 6 feet. But in some very great buildings they are made longer.

As to the form, there are various kinds, denominated in general by *streight stairs* and *winding stairs*. The streight stairs always fly, but never wind; and are of several kinds, as

Direct fliers, these go directly from one floor to another, without turning to the right or left, seldom used but for cellars or garrets.

Square fliers, these fly round the sides of a square newel; and at every corner of the newel there is a square half-space, after which the next flight goes off at right angles to the former. A *newel* is the upright post, about which a pair of stairs turns, being placed in the middle of the stair-case. Sometimes the newel is a solid pillar, and sometimes an open space.

Triangular fliers, these fly about the sides of a triangular newel. So every flight goes off from the last in an angle of 120 degrees, or 60°, reckoning backwards; and at each angle is a half space. All stairs in general that go directly forward, are called *fliers*.

French fliers, are those that fly first directly forward to the half-space; and then directly backward again to the top, or to another half-space; and the next flight (if there be any more) goes over the first &c. These stairs are in common use.

Winding stairs, are those that never go streight but continually change the direction. Of the

Some wind round about a circle, some about an ellipse, others about a square or a triangle. And each sort may either go about a solid newel or an open one. In all winding stairs, the steps are broader at one end than the other.

Circular winding stairs, are those which go about a solid newel; here the edge of each step is a right line pointing to the center of the newel. They are often found in church-steeples and cathedrals. The steps are of the same form, if the newel be open, as in the monument at *London*. In some sorts of these, the foreside of each step is circular or elliptical, according to the different fancies of people.

All kinds of winding stairs take up less room than square ones, or any others.

Where there is much room, the length of a step must be equal to the diameter of the newel. In other cases make the diameter of the newel $\frac{1}{4}$, $\frac{1}{5}$, or $\frac{1}{6}$ the diameter of the whole stair-case, according as the stair-case is great or small. If the newel be open, make the newel half the diameter of the stair-case, to be more enlightened. In very narrow places, these stairs may be made without a newel.

Elliptical winding stairs, are divided like the round; here all the windows and doors are in the middle and head of the oval. The newel here is also an ellipse.

Stairs may also wind round a square or a triangle, and many other ways may be invented.

Double winding stairs are so contrived, that two persons, one ascending and the other descending, shall not come at one another. The foot of one of these stair-cases is opposite to the foot of the other, and they make their ascent on opposite sides. The newel in the middle is hollow to convey light, and very large.

Fig. *Quadruple winding stairs*; there are four stair-cases which have four entrances, and go up one over the other, and being made in the middle of a building, they lead to four chambers; so that the inhabitants of one need not come upon the stairs of the others. This must have a very large newel which is open, to give light. They all see one another go up and down, without hindering one another.

In square stair-cases, when you are forced to place steps in the angles for want of room; you cannot place more than three steps at an angle, to wind when there is but 6 or 7 feet in breadth. For 9 feet broad, take 4 steps at an angle. And for 10 feet width, take 5 steps. These sorts are called *mixed stairs*, *dog-legged stairs*, and *stiers and winders*.

There are a sort of stone-stairs, going up in towers, which wind; and every step has one end in the wall, and is supported by the lower step where the foreside of the upper step lies upon the backside of the under step, all the way.

The landing-place of any stairs, or the space between the wall and the steps, must be in breadth the length of a step, and at least a fourth part more.

To find the number of stairs.

Having pitched upon the height of a step; divide the height of the room (in inches) by the height of a step; the quotient is the number of steps.

To find the breadth of a step, divide the width or space (measured on the ground) by the number of stairs; the quotient shews the breadth of a step.

SCHOL.

To find the twist of a rail in a stair-case. Suppose the section of the rail to be a square or rectangle, and the newel to be circular, and that the

stairs turn round in a circle. Let a be the height of a step, b the breadth; r the radius of the newel, or the distance from the center to the inside of the rail, R the distance to the outside. Then the radius of curvature of any part of the rail on the inside will be $\frac{aa+bb}{bb} r$; on the outside $\frac{aa+bb}{bb} R$.

Let ABC be a step, AB (b) the breadth, BC (a) the height, A = degrees in the angle BAC, c = hypotenuse AC, $c = 3.1416$. Then the twist in a quadrant will be A, and the twist in every foot of length of the rail is $\frac{2b}{cdr} A$, in degrees;

taken on the inside. But it is the same in any side, as the Section is every where the same. And the twisting lies from the newel, in the ascent.

The upper side in the ascent is concave, and therefore leaves the plain (of an ellipsis) which touches it at first. But the deviation is so small, that it may be neglected. The length of a quadrant of the rail is $\frac{cdr}{2b}$. And the deviation in $\frac{1}{16}$

part of this length is but $\frac{.000645ar}{d}$. The prac-

tice may be thus, take a piece of timber as dDHb, and with a line of $\frac{dd}{bb} r$ feet for the radius, describe

with chalk the arch bfd on the inside, and with the radius $\frac{dd}{bb} R$, the arch HFD; the part between

these arches is the rail. Then since the twist affects every side of the rail alike; lay it down on a plain floor, so as the upper surface at D be parallel to the horizon, and make DE, EF, FG, GH, each a foot; the length of a quadrant of the

Fig. 62. rail being $\frac{dcr}{2b}$; and drawing Ee, Ff, Gg, Hh, &c.

perpendicular to the side DH; place an instrument to the angles, first upon Dd which will be level; and then upon Ee, Ff, &c. Now the upper side is to be so formed or cut away, that the instrument placed upon Ee will lean to the right

hand $\frac{b}{1.57dr}$ A; upon Ff, $\frac{2b}{1.57dr}$ A; upon Gg, $\frac{3b}{1.57dr}$ A, and so on. And the other sides must

be cut square to it, in every place. And the top side must be a very little concave.

PROP. IX.

To describe the several methods of roofing.

The insides of roofs are made of timber; and on the outside they are covered with tiles, slates, lead, straw or ling; and the form of the roof will be different for each of these. There are two sorts of roofs, *plain roofs* and *hip roofs*; the plain roofs have gable ends, and the hip roofs have none.

The wood work of a plain roof is made thus if a room be above 12 feet long, there is placed in the middle of it a pair of files or *suspars*, called the *principal rafters*, on the top is laid the *rigging tree*, which lies horizontally with its ends on the tops of the gable-ends of the room, and its middle on the suspars. On the sides are laid the *ribs* or *purlins*, one or two on a side, according to the wideness of the room; these are to nail the spars or rafters to, which are to be laid all along, at about a foot and half distance. Then the *lats* are nailed cross the spars at about 11 inches distance for tiles. In short rooms there is no occasion for

suspars

spar. Suspars ought not to lie over the heads of Fig, windows or doors.

For the *pitch*; let AB be the breadth of the 63.

house, make AC and CB = $\frac{3}{4}$ AB, to meet at C, then that is called the *true pitch*. And when the gable-end ACB is built; two spars or rafters AC and BC, of a just length, are nailed together at the top C, and set upon the walls at A and B; and the gable-end is built by these. Here the

perp. CD is = $\frac{AB}{4} \sqrt{5}$. But it is now very com-

mon to make roofs under pitch, so as the angle ACB may be a right angle; and then the perpendicular CD is equal to AD, or half AB, and it is called the *square pitch*. No roof should be made lower than square, for then it will rain in. Here AC is the *riggin-tree*; FF, FF the ribs. In the square pitch AC is a mean proportional between

AD and AB, and therefore $AC = AB \times \frac{\sqrt{2}}{2} =$

$\frac{AB}{\sqrt{2}}$. In some cases they make the pitch, so that

AC = AB, which is very high.

For hip roofs. Let LA be the top of the side 64.

wall (or you may lay it out on the ground), CN the *riggin-tree*; AB the breadth of the house, AD = DB, and CD a perpendicular, then the plane ACB will be perpendicular to the horizon.

And ACB will represent a gable-end, instead of which the hip-roof is to be made, and the hips must meet at the point C. Produce LA to G, so that AG = AD, and draw GK parallel to AB; and make the two squares GADH, and HDBK. Then taking AC of a proper length, GC and AC will be the two hips, whose length will be

$\sqrt{AC^2 + AG^2}$, or $\sqrt{AC^2 + \frac{1}{4} AB^2}$. AB is a

strong *dormant*, or *summer*, going cross the house, GD, KD the *dragon beams*, for the hips to stand

on,

Fig. on, being framed into the beam AB at D. The ribs are to lie along upon NL, CA, and CG and other cross ribs are to lie upon CG and CH upon all these the spars are to be nailed. The length of CH is equal to CA. The *back* of the hip is the angle made by the two planes CGL and CGK. The planes CGD, CKD are perpendicular to the horizon, both passing through the perpendicular CD.

The practice is much the same for *bevel* roofs. A *bevel house* or roof is one where the ends are not perpendicular to the sides. As suppose the angle LAB is an oblique angle; then GADH and KBD are two rhombus's. And the length of GC = $\sqrt{GD^2 + CD^2}$, and KC = $\sqrt{KD^2 + CD^2}$.

Tiling. Of tiles there are various sorts, as common or pan tiles, plain tiles, riggin tiles, gutter tiles, Flemish tiles, hip tiles, Dutch tiles, &c. In covering a house with pan tiles, one must be very exact in the gage for placing the lats. For if the gage be too short, the tiles will want lap on the sides, which is a great fault. And if it be too long, they will want lap on the tops, but this is not so material. Flat gutters must be laid with lead, and so should all gutters, but sloping gutters are often laid with gutter tiles, and sometimes with common tiles. Tiles laid on a house-side are laid without lime, and pointed on the inside with lime mixt with hair. The riggin is covered with riggin tiles, made hollow on one side, and broader than the other sort. But workmen often make shift with common tiles, cutting off the roll, then they are laid in mortar. A short length sometimes happens at the bottom of the roof; and then the workmen cuts away the shoulder of the tiles, then they will fit into their places. In hip roofs, the hips are covered with hip tiles, which are much like riggin tiles. Hip tiles have a hole in one end

to nail them on by. Some houses are covered with Fig. plain tiles, which hang by pins, in the nature of slates.

Dutch tiles are only used instead of chimney corner stones.

In flat gutters that are laid with lead, take care that the gutter boards have a good descent. If one sheet of lead does not reach the length, the plumbers folder another to it, which often breaks in the joint, and lets in water. A better way is to put the end of the second sheet under the end of the first, making a proper descent for it in that place.

Slating. Slates are flat stones like flags, raised out of a quarry, and are an inch or two thick, according to the breadth; they are dressed into the form of a parallelogram, of different sizes from 3 feet to 1 foot. The timber here must be stronger than for tiles, and the lats must be oak, and laid wider at the bottom than top of the roof, for the biggest slates come on first. The slates have one or two holes in the top. The roof is made hollow, or pretty sharp at the top; otherwise the slates would not lie, but ride upon the heads of one another. Tiles are laid in rows upward from the bottom to the top of the roof; but slates are laid in horizontal rows, beginning at the bottom with courses of the largest slates. They are all laid in moss, and hang on the lats, by bones or oak pins put through the holes. As you rise higher, the courses decrease gradually to the top; they must not be laid joint upon joint, the slates in the top course are toothed or notched on both sides, and the riffin slates of one side of the roof fixt into those on the other side, so that they cannot blow off. They are pointed with mortar for 2 or 3 courses at top and bottom. The roof is 3 tile thick at every point, if right laid. To try the goodness of slates, hang them by an edge in your fingers,

Fig. fingers, and strike against them with a stone or with a finger; if they ring they are good; but if they chatter, they are soft, or cracked in some place, and so are useless. The same way tiles are tried.

Leading. When houses or churches are covered with lead, which is very chargeable; large domes of oak are laid cross the building at 9 or 10 feet distance, made highest in the middle; and on these, as many ribs as are sufficient, and spars nailed to these, all of oak; and instead of laths thin oaken boards are laid close, upon the spars, and nailed down, like a floor. Then the sheets of lead are laid close by one another, reaching from bottom to top. And the edges are turned over one another, and beat down round, inclosing one another so that no rain can get in at the joint. When a sheet is too short, another sheet is laid on below and its end put 3 or 4 inches under the end of the first. They must be nailed down in several places with short nails with broad heads.

Straw is another covering. It is laid by a pool of water for some days, and is constantly watered. Then it is drawn into sheaves, and led away, and must lie for a month at least, before it is thatched. When it is thatched, there is first made a deal of bundles of dry straw, which are laid all the way cross the spars, and tied down with ropes of straw, going round the ribs and lats. Then the *thatcher* begins to thatch at the bottom, and carries up a course of thatch, two feet wide, to the top; and at the top lays some mortar upon it. The manner of thatching is this, the workman has a shaft of thatch brought up; he takes a part of it and makes a wisp of it, tying the top fast together with part of the straw, to keep it together; then he thrusts his instrument (the sprittle) into the dry bundles of straw (if it is a new house, or into the old covering, if it is an old one), to make a hole, into which

which he thrusts the wisp with the head foremost, Fig.

with the sprittle, as far as he can get it. Then he puts in another by the side of it, and another by the side of that as close as he can squeeze them in, till he has got the breadth of his course; then he works in another cross row of wisps, and above that another, and so on till he comes at the top. The ladder he stands on to work, terminates the breadth of each course. As soon as each course is thatched, he takes a switching knife and strikes off the top of the course, cutting it smooth all the way down, and cuts the eaves streight at the bottom. In some places they thatch with reeds instead of straw, which lasts many years. Thatching with straw requires a sharper roof than tiles.

Ling. Here the pitch must be very great, and very strong timber to bear a coat of ling. Ling is thatched by only laying one course upon another, horizontally, all the length of the house; it must be laid sloping, with the top outward, this is no more than making a hay stack; but it ought to be bound on here and there with ropes, or fastened with belks. To make the riggin, a number of fods is cut of a proper length, and laid cross, all along the riggin, and spelked or pinned down. One of these roofs requires a great deal of ling, and rises very high.

Lead is the best cover, but is immensely dear; but for common houses, slates are quite the best covering; for they are neither hot in summer nor cold in winter; but tiles are both. Neither are they so subject to rain in as tiles. A slate roof is easily repaired, they only require stopping with moss once in 5 or 6 years. A thatched house is far warmer than a tiled house; but then thatch does not last long; and is subject to be blown off in high winds. Tiled roofs are so cold in winter, that most people are obliged to ceil them on the inside.

And

Fig. And many people ceil the top of a room, on the underfide of the joists, and sometimes on the boards. In a garret there are joists laid on purpose, called *ceiling joists*. Ceiling of rooms makes them more lightfome, and lessens the noise above head. To ceil a room, there must be ceiling lats nailed all along on the underfide of the joists, as rank as may be, and these are to be plaistered over with lime, made up with hair; as all plaistering is made.

PROP. X.

To explain the several orders of architecture.

In the making of columns, there are five different forms and proportions, called the five orders: the *Tuscan*, *Doric*, *Ionic*, *Corinthian* and *Composite*. A column consists of three parts, the *pedestal*, the *column or body*, and the *entablement*. The parts of the pedestal are the *base*, *die*, and *cornice*. The parts of a column are the *base*, *shaft*, and *capital*. The parts of the entablement are the *architrave*, *frieze*, and *cornice*. These and their parts are called *ornaments* or *mouldings*. All their parts are measured, and set off by *modules*. A *module* is the diameter of a column taken at the base, and is divided into 60 parts or minutes. ABC is a column.

65. A the pedestal, B the shaft, C the capital, above is the entablement. Every column must be smaller towards the top than at bottom, taking the example from the boles of trees that taper all the way up. A column is said to be more beautiful, if it swell a little about the middle, or rather at about a third part of its height.

The capitals of the Tuscan and Doric are plain; that of the Ionic has a *scrol* or *volute* upon it; the Corinthian and Composite are adorned with flowers and the leaves of trees. The height of the entablement

ment, or of the architrave, frieze and cornish, Fig.
 $\frac{1}{2}$ the height of the column, in the Tuscan and
 Ionic; and $\frac{1}{3}$ the height in the Ionic, Corinthian
 and Composite.

An account of the ornaments which are to be
 used in some or other of the orders; going up-
 wards from the top of the column.

The *capital* is the head of the column; near the
 top of the column goes a round list.

Architrave, the moulding above the capital,
 made upon a stone or beam of wood which may go
 over several columns.

Annulet, fillet or list, a narrow flat moulding
 which goes round the column.

Altragal, a little roundish moulding mostly at the
 top of a column.

Frieze, a large flat member above the architrave.

Echine, or *ovola*, a part above the frieze, a quar-
 ter round on the lower part.

Voluta, an ornament that represents the barks of
 trees twisted and wreathed, in the Corinthian order.

Abacus, a square table on the top of a pillar.

Cimace, scima recta, or *ogee*, a member consist-
 ing of a round below, and a hollow above, like
 the letter S.

Scima reversa, an ogee turned upside down.

Fasciæ, broad lists, belts, or fillets, commonly
 used in the architrave.

Fryglisse, part of a frieze.

Cavetto, or *scatia*, a hollow moulding.

Crown, the finishing part of an ornament.

At the bottom of the column are,

Annulets, or flat rings.

Altragal, as before.

Scotia, or *cavetto*.

Torus, a round member going about the base.

Plinth, a square part, the lowest part of the base.

Cimace,

Fig. Cimace, as above.

Die, a square part included between the base and cornice.

A Table shewing the length of the principal parts in the five orders, in modules and parts.

	body		pedest.		base		capital		archit.	frieze	corn.
	m.	p.	m.	p.	m.	p.	m.	p.			
Tuscan	6	15	2	0	0	30		30	30	30	35
Doric	7	30	2	35		30		30	30	45	40
Ionic	8	02	50		35		22		36	32	48
Corinth.	8	10	3	0	30	1	10		37	35	0
Compof.	8	22	3	25	30	1	8		45	42	50

The diameters of the columns at the top, are to be as follows, the *Tuscan* $\frac{3}{4}$ the diameter at bottom. The *Doric* $\frac{4}{5}$; the *Ionic* $\frac{5}{6}$; the *Corinthian* $\frac{6}{7}$; the *Composite* $\frac{7}{8}$, the diameter below.

The *Tuscan* column is the strongest of all the orders; the next the *Doric*, then the *Ionic*, the *Corinthian*, the *Composite*. And in building, the strongest must be set lowest. Therefore the *Doric* is placed under the *Ionic*; the *Ionic* under the *Corinthian*, &c. The *Tuscan* is seldom used but under ground.

The inter-columns, that is, the spaces between the columns, are thus regulated. In the *Tuscan* that space must be 4 diameters at the base, in the *Doric* 3; in the *Ionic* 2; the *Corinthian* $2\frac{1}{2}$, *Composite* $1\frac{1}{2}$ diameters below. The base and capital of any column ought to project, beyond the body $\frac{1}{3}$ or $\frac{1}{6}$ the diameter at bottom.

Columns placed in any front of a house must be all of the same sort, of the same bigness and height and at the same distance, except where there are doors, and there they must be wider, as in the middle of the front, where the main door is generally placed. Pillars are frequently placed under piazzas (or P. Hes) and cloysters.

Pilaster

Pilasters, or half pillars, are those that usually stand behind columns for arches to stand on; the whole appear like double or clud columns. They are often made to jut out of a wall, about a third part of their diameter. When they adjoin to a column, they have the very same dimensions, capital and base with the column, according to its particular order.

The height of the pedestal of any column must be a third part of the whole column; or more particularly, $\frac{2}{3}$ for the *Tuscan*; $\frac{1}{4}$ for the *Doric*; $\frac{2}{3}$ for the *Ionic*; $\frac{2}{3}$ for the *Corinthian*; and $\frac{3}{4}$ for the *Composite*. The height of the capital $\frac{1}{4}$ diameter, in the *Tuscan*, *Doric*, and *Ionic*. And $\frac{1}{2}$ diameter in the *Corinthian* and *Composite*.

The *Persian order* of architecture, is to have the figures of men and women instead of columns, to support any building. There is also the *Attic order*, where small pillars, &c. are placed upon water works; and the *Gothic order*, which is a strong and coarse manner of building; and the arches are often carved with the figures of thistles, nettles, &c. And Gothic arches are those that consist of two arches, making an angle at the top.

Arches are often made to stand upon pillars as in all churches; and in bridges, where the piers are made exceeding strong, to resist the water.

Columns are often made *fluted*, that is, with semicircular hollows running along the body of the column, from the base to the top, 24 in number. In the *Doric*, there is no space between; in the rest, a flut runs between the flutes.

A draught of the several parts or ornaments will give a better notion, than a verbal description; as follows. Aa scima recta, Bb scima inversa, Cc the crown, Dd torus, Ee cavetto, all these belong to the cornice. Then Ff is the freeze, Gg the architrave. In the capital, Hh the cimatum, Ii the

A a

abacus,

Fig. abacus, Kk echinus, Ll astragal. M the body of the column.

67. In the base, M the body of the column, Aa the pedestal, Bb torus, Cc cavetto, Dd torus, Ee plinth. Ff cymacium and lists, Gg cavetto, the square, I base.

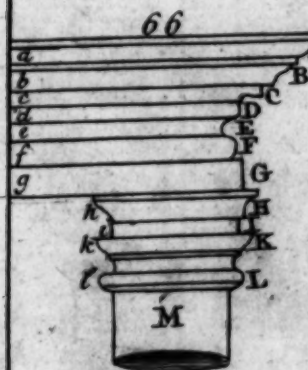
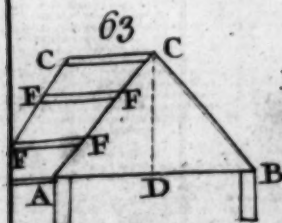
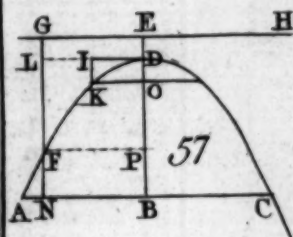
I shall not trouble the reader with the measure of these parts, as almost all authors differ therefrom one another; for as they are all designed for beauty and ornament, different people have different notions of what is beautiful; some people thinking them prettier in one form and some in another. And the same observation may be made concerning the stated lengths and thickness of columns, for all these canonical proportions are merely arbitrary, and given them for the sake of beauty. But true mechanical beauty consists in giving a due proportion of strength to every thing, and to every part of it. And the bigness of columns ought to depend upon the hardness and strength of the matter they are made of. If a solid pillar of brass or iron was to be made, of the stated height of any of these orders designed for stone it is evident, it need not have its diameter so great to bear the same weight. And if a column was to be made of the softest sort of stone, its diameter would require to be made much greater, or else it would be crushed to pieces with the incumbent weight. So that it is absolutely necessary that the dimensions must depend upon the strength and goodness of the materials; come what will of beauty, which is nothing but fancy.

SCHOL.

It may not be amiss here to shew the investigation of two curious mechanical Problems, concerning the strength and pressure of walls and pillars.

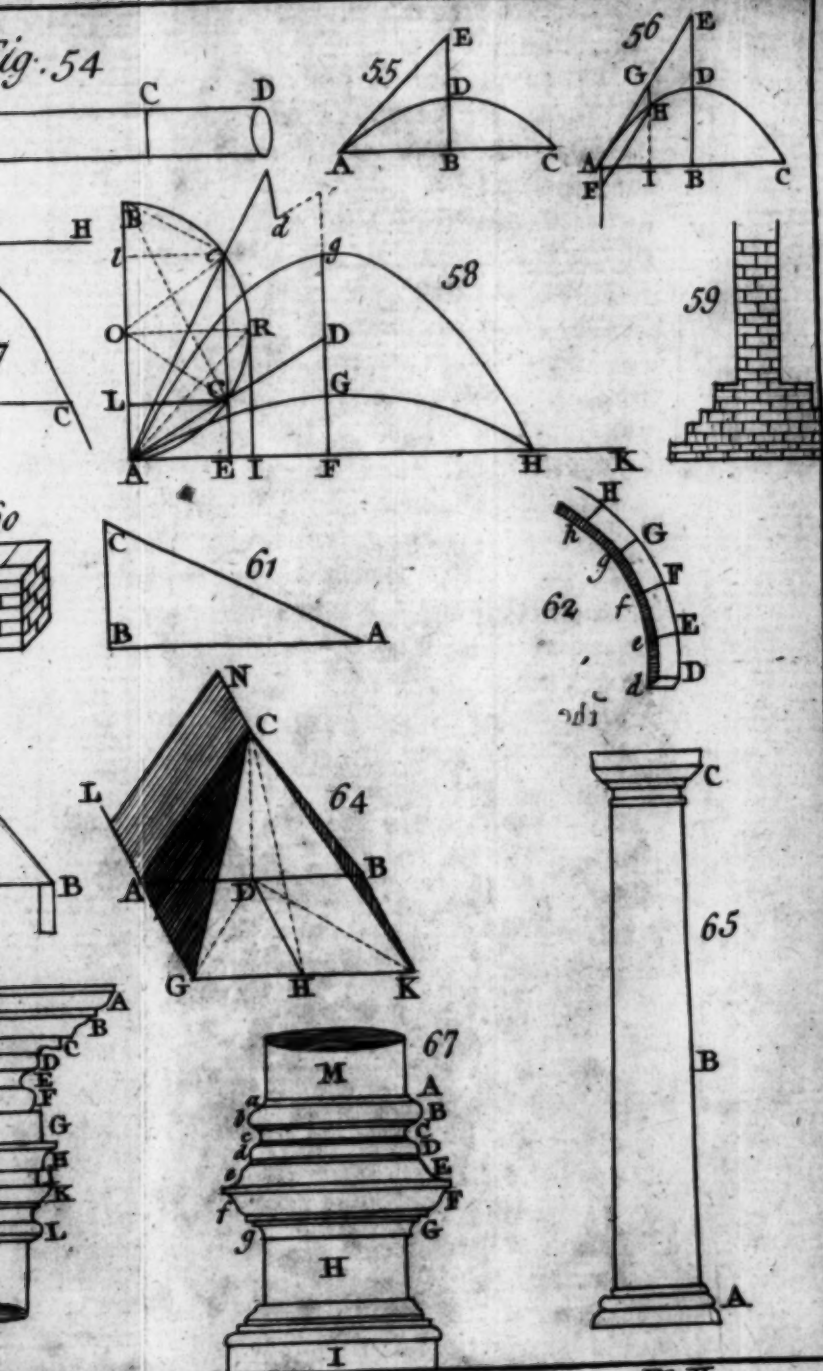
Fig. 54

A B



Miscellanies

Fig. 54



Prob. 1. To find the nature of the curve AEB or Fig. 68.
 AC, in the form of which, a wall of a given
 thickness being built, it shall be of equal strength every
 where to sustain the weight; or shall not be sooner
 broken to pieces in one part than another.

Let $AD = x$, $DE = y$, the thickness $= 1$;
 then the fluxion of the figure is $y\dot{x}$, and the weight
 is $F! : y\dot{x}$. Also the strength at D is as DE or
 therefore $F! : y\dot{x}$ is as y , or $F! : y\dot{x} = ry$, and
 finding the fluxions $y\dot{x} = r\dot{y}$, and $\dot{x} = \frac{r\dot{y}}{y}$, and

is $r \log. y$, which denotes the logarithmic curve,
 whose asymptote is AL. Therefore if a wall ABC
 be built, whose two sides AB, AC, are in the form
 of the logarithmic curve, of infinite length AL,
 it will no sooner be crushed to pieces in one part
 than another.

Cor. If the wall EGHF be made upon EF, whose
 height EG is the subtangent to the curve; and the
 part AEF taken away, the part BEFC will still be
 of equal strength every where.

For (Cor. 1. Prop. IV. log. curve) the area
 AEF = GEFH, and the weights equal.

Prob. 2. To find the nature of the curve AEB,
 being turned round its axis AL, shall generate a
 pillar, that shall be of equal strength in every part.

Let $AD = x$, $DE = y$, $c = 3.1416$; then the
 surface of the solid is $cyy\dot{x}$, and the solidity or
 weight, as $F! : cyy\dot{x}$. Also the strength at D is as
 therefore by the question, $F! : cyy\dot{x}$ is as cyy ,
 therefore $F! : cyy\dot{x} = ryy$, in fluxions $cyy\dot{x} = 2rcy\dot{y}$, and
 $\dot{x} = \frac{2r\dot{y}}{y}$, and $x = 2r \log. y$, whence AEB is a lo-

garithmic curve, as before, and its asymptote AL.
 Therefore a solid pillar, in form of the logarithmic

Fig. mic curve of infinite length, will no sooner be crushed to pieces in one part than another.

Cor. If $EG =$ half the subtangent, and EGH be a cylinder placed upon EF , instead of the solid AEF . Then the part $BEFG$ of the pillar will be every where of equal strength.

For (Prop. V. log. curve) the solid $AEF =$ cylinder $EGHF$, and therefore their weights are equal.

And hence it also follows, that if the wall or pillar be made of any glutinous or tenacious matter, and turned upside down, and suspended at the thick end in the air; its weight will no sooner pull it in pieces in one part than in another.

PROP. XI.

To shew the situation and magnitude of garden courts, gates, woods, &c.

No house should be built too far from the street; that is, too far back, if it stand in a town; for it causes a great deal of labour going back and forth to the house, and nothing can be seen in the town. And when it happens that a house has been injudiciously placed too far back, this fault may in part be remedied, by the loss of a little ground; and that is, by taking away the segment of a circle from the yard or court, and let it lie to the street.

An area or court before a house ought to be as broad as the house, and must be a square or rectangle; if a rectangle, the length must be the breadth and half, but not more. But towards the street, it may sometimes be terminated by the end of a circle or ellipsis. And no trees ought to be placed in it, but only shrubs, which one can

over. Gravel walks must be laid, streight to the door, and round about, and borders of flowers planted to beautify it. The rest to be grass.

Gardens must be made large enough to grow such fruits and vegetables as are sufficient for the family. A garden ought to join to the south side, or the warmest side of a house, which will be very pleasant and beautiful for most part of the year; being placed in a warm situation, they will be the better defended from the cold blasts of the spring, that oftentimes destroy all fruit. The north and north-east winds do endless mischief, and must be guarded against. An orchard may stand at a distance from a house, but if it have a warm situation it will be better. It is commonly placed on the backside of a building, whatever way that happens to be.

Large trees may be planted at some distance from the house, to defend it from cold and stormy weather. But by no means let them be placed so as to hinder a good prospect; and a prospect is the better, as it is the more extensive, and consists of more variety of objects. Hills and dales, and a mixture of land and water are very pretty, as are woods and groves, at a distance. A house is best defended by trees, when they are placed towards the south-west or north-east; for the strongest winds come from the former, and the cold weather from the latter; and all these circumstances ought to be considered.

Gates for foot people to pass through should not be less than four feet wide; nor less than eight for carriages. And in large buildings, they may be 10 or 12 feet wide, but not more. The principal gate must face the main door of the house; a fair entrance gives a great grace to a building. Some say the height of a gate should be once and

A a 3

a half

Fig. a half or twice the width, which in large gates certainly more than is needful. The gate should be lowest in the middle; and the top, of a circular form. Ornaments are often placed over the tops of gates and out doors, which makes them look very pretty.

Many more particulars ought to be taken notice of, which, for scarcity of room, I am forced to omit. But these I have treated on are the principal.

A R T. XV.

M U S I C.

MY design here is to treat of this delightful art, so far as it depends on mathematical and philosophical principles. I have indeed treated upon some parts of this subject in other books, and have partly anticipated the subject. But I shall make abstracts of these things, as I have occasion, for the further prosecution of this matter. The practical part of music is well known, and in great perfection; but the speculative part is but known to few, and is what I design to treat of in this paper.

D E F I N I T I O N S.

Def. 1. *Music* is the science of pleasing sounds, and consists of melody and harmony.

Def. 2. *Melody* is the pleasure arising from hearing a succession of musical sounds, as in any tune or piece of music; this relates only to a single part, and is the same as the *air* of a tune.

Def. 3. *Harmony* is the pleasure arising upon hearing a composition of musical sounds in conjunction, or music in several parts.

Def. 4. A *note* or *tone*, these two words are used promiscuously either to signify the *degree* of sound, which it stands in the scale; or the *distance* between two of them.

Def. 5. The *scale* or *gamut* is a list or catalogue of all the notes in music, placed in their proper order.

Fig. Def. 6. *Pitch* is the height to which the scale is set; or the voice or any instrument.

Def. 7. *Key* or *key note*, is the lowest or ending note of a piece of music. There are two of these, the *flat key* and the *sharp key*.

Def. 8. *Cliffs* or *clefs* are marks set at the beginning of every piece of music, to denote what part it is, whether treble, base, &c. which save the labour of writing it down.

Def. 9. *Moods*, certain marks used to denote the various measures of time.

Def. 10. *Cadences* are the concords and discords taken at the close or conclusion of a piece of music, or of some part of it; by these the music naturally falls into, and terminates in the key.

Def. 11. *Cords* or *Concords*, are such notes as produce a harmony, when they sound together. Discords produce none.

Def. 12. *Mudulation*, is the art of passing out of one key into another.

Def. 13. *Transition*, is breaking a note into several lesser notes, to sweeten the roughness of a leap, by gradually passing from that note to the next.

P R O P. I.

In any musical string, whose diameter = d , length = l , weight = w , its tension or weight that stretches it = T , and $f = 16.1$ feet; then the time of its vibration will be $\sqrt{\frac{wl}{2Tf}}$ in seconds.

This is demonstrated in Prob. II. Sect. 3. Fluctuations, to which I must refer the reader, for the proof of it.

Cor. 1. *The number of vibrations performed in a second, is* $\sqrt{\frac{2Tf}{wl}}$.

Cor. 2. *The square of the time of vibration of any musical string is as the length and weight directly, and tension reciprocally.*

Cor. 3. *If the matter is the same; the times of vibration of different musical strings, are as the diameters and lengths directly, and the square root of the tensions reciprocally.*

For w is as ddl , and therefore the time is as $(\sqrt{\frac{ddl}{T}} =) \frac{dl}{\sqrt{T}}$.

Cor. 4. *If the matter is the same, the number of vibrations in a second, of different strings, are as* $\frac{\sqrt{T}}{dl}$.

Cor. 5. *In the same string and tension, the time of vibration will be as the length.*

Cor. 6. *In the same string and tension, the number of vibrations in a second, will be reciprocally as the length.*

Cor. 7. *If the matter be the same, and the tensions as the squares of the diameters; the times of vibration will be as the lengths.*

Cor. 8. *If the matter and the length be the same; and the tensions as the squares of the diameters; the times of vibrations will be equal; and they will give the same sound.*

S C H O L.

By this last Corol. the strings ought to bear the same pitch, and both break together. But I always found the thickest to break first, and the smallest to bear the highest pitch.

It

Fig. It may not be amiss here to relate some experiments I made of the vibrations of strings.

I took a wire string 57.95 inches long, weighing $87\frac{2}{3}$ grains, which fastened to a virginal, and strained with 7lb. weight; a quarter of it was unison with EE flat in the base. By this theory it made 65 vibrations in a second; and therefore 4×65 or 260 is the number of vibrations which the string E flat (in the mean) makes. But this instrument was a note below pitch.

A string 52.9 inches long, weight 71 grains strained with $10\frac{1}{2}$ lb. on a spinet, half of it was unison with AA in the base, whence the whole made 86.7 vibrations in a second. And 4×86.7 or 347 is the number of vibrations which A (high in the base) makes in a second, and E flat makes 245.

At 7lb. weight it was with F, which makes 288 vibrations, whence A would make 353, and E flat 250.

At 9lb. weight it was with G, which makes 324 vibrations, whence A would make 362, and E flat 255.

By a former experiment E flat was 254. Therefore at a mean of 5 experiments, E flat made 254 vibrations, and C below made 212.7.

P R O P. II.

If two musical strings be made to vibrate and sound together; the sweetness of the harmony will be greater or less, according as the coincidence of their vibrations are many or few. And if their vibrations never coincide, they produce the most harsh and disagreeable discord.

69. These things are matter of experience. For let AB, ab be two strings of the same matter and bigness for length and breadth, and stretched with the same

same weight, and made to vibrate; their vibrations will be compleated in the same time, by Prop. I. 59. and they will both give the same sound, whether they sound together or separate. Here their vibrations always coincide, and their united sound is *unison*, which is the most perfect harmony that can be.

Now let one of the strings *ab* be stopt in the middle at *c*, and *AB* and *cb* be made to vibrate; then (Cor. 4. Prop. I.) *cb* will vibrate just twice as fast as *AB*; and their vibrations will coincide at every vibration of *AB*, and at every other of *cb*; and this is the sweetest harmony that can be made by strings of different lengths, and is called an *octave or eight*. And there is so much likeness in the idea or perception of these two sounds, that musicians do not scruple to account all eights the same. In the same manner if *cb* was stopt in the middle, and made to vibrate with *AB*, it would vibrate four times as fast, and every 4th of its vibrations would coincide with every vibration of *AB*; and the same harmony, and the same similitude, will still remain.

Again, let *ad* be a third part of *ab*; and the string being stopt at *d*, let *AB* and *db* vibrate together, then every third vibration of *db* will coincide with every second of *AB*; and produce a delightful harmony, which is the next agreeable concord, and is called a *fifth*.

In like manner if *af* be a fourth part of *ab*; and *AB*, *fb* vibrate together; their vibrations will coincide at every fourth vibration of *fb*, and every third of *AB*; which produces that agreeable harmony called a *fourth*.

After the same manner, if *ag* be a fifth part, and *ab* a sixth part of *ab*; and *gb* or *bb* be made to vibrate with *AB*; then the fifth vibration of *gb* will always coincide with the fourth of *AB*; and the

Fig. the sixth vibration of bb , with the fifth of AB and they will still produce a delightful harmony but less and less perfect. These are called the *greater and lesser thirds*.

But let a part of ab be stopt, which is incommensurable to AB , the sound of these, vibrating together, will produce nothing but a disagreeable noise, without the least harmony. So it is evident that it is the coincidence of the vibrations that causes the harmony; and the more frequent the coincidence is, the greater is the harmony; and where there is no coincidence, there is no harmony at all.

70. Cor. 1. *If there be four strings of equal size and tension, whose lengths are ba , bg , bd , bc ; and are made to sound together, they will produce the most agreeable harmony, that is possible to be made with four strings, in the compass of an octave.*

For if the length ba be 1; bg will be $\frac{4}{3}$; bd , $\frac{2}{3}$ and bc , $\frac{1}{2}$. Therefore 4 vibrations of ba are equal to 5 of bg , and to 6 of bd ; and to 8 of bc .

Cor. 2. *And the greatest harmony that can be made with 6 such strings, will be when their lengths are 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$.*

For the number of vibrations will be reciprocal as their lengths; whence 1 vibration of the first will be equal 2, 3, 4, 5, 6, vibrations of the 2^d, 3^d, 4th, 5th, 6th, respectively. And after these periods they all constantly coincide.

Cor. 3. *If several parts of the string ab (which do not note any cords) be multiplied together; the product will denote the cord, which is the sum of all the rest. Or they may be discords.*

For if $gb = \frac{8}{9}$, and $db = \frac{3}{4}$, then $\frac{8}{9} \times \frac{3}{4} = \frac{2}{3}$ that is, a whole tone and a fourth, make a fifth. For the string gb , stopt at d , produces a fourth and above

above g , or a fifth above a . In like manner $\frac{5}{2} \times \frac{4}{3}$ Fig. $\times \frac{3}{2} = \frac{10}{3}$, that is a lesser third, a greater third, and a fourth, make an eight. And so of others.

P R O P. III.

If ZA be any musical string; and if ZA , Zb , Zc , 71 . Zd , Ze , &c. be taken in geometrical progression; and if ZA be made to vibrate, and then stopt successively at b , c , d , e , f , &c. The sounds produced thereby will ascend equally, or rise by equal degrees.

That the sound of any musical string is caused by the vibrating motion thereof striking the air, is matter of experience; and likewise the more any string is stretched, the faster it vibrates, and the acuter the sound will be. And when the stretching force or tension is the same; the shorter the string is, the more acute the note will be, and the vibrations quicker.

It is also known by experiments, that in any string whatever, if it be stopt in the middle, it will then sound an eighth higher. If a third part be stopt, it will sound a fifth higher. If a fourth part be stopt, it will sound a fourth higher. If a fifth part be stopt, it will sound higher by a greater third. If a sixth part be stopt, it will sound higher by a lesser third; and so on, whatever be the length of the string: for the acuteness of the tone depends upon the quickness of the vibrations. Therefore if AZ , bZ , be two strings, or one string stopt at b , and if Ab be some determined part of AB (as $\frac{1}{2}$); then the sound of bZ is raised a determined degree (a lesser third): for the same reason if bZ be a string, and bc the same determined part of bZ ; then stopping it at c , will also raise the sound by the same degree. And if cZ be a string, and cd still the same determined part of cZ ; stop-
ping

Fig. ping at d will also raise the sound an equal degree higher; and so on. Now Ab is the same part of AZ , as bc is of bZ , and as cd is of cZ , &c. because $ZA, Zb, Zc, Zd, \&c.$ are in geometric progression, and $Ab, bc, cd, \&c.$ in the same geometrical progression. Therefore the string AZ stopt at $b, c, d, e, \&c.$ will raise the sound gradually by equal degrees.

Cor. 1. Hence $\frac{Ab}{AZ} = \frac{bc}{bZ} = \frac{cd}{cZ} \&c.$ and each of them may denote that given degree, the sound raised, or elevated in stopping at $b, c, d, \&c.$

Cor. 2. And consequently, when any part as Ad but small, the height the sound is raised by the string AZ, dZ , is nearly as $\frac{Ad}{AZ}$.

For then Ab, bc, cd , will be nearly equal. But that degree of ascent will be more exact

$$\frac{Ad}{\frac{1}{2}AZ + dZ}$$

Cor. 3. If $Ab, bc, cd, de, \&c.$ be taken extremely small; any part of the string as Ag will be analogous to the logarithmic area; and the rise of sound, the logarithm corresponding.

For $Ab, bc, cd, \&c.$ being in geometrical progression, and the correspondent ascents of sound equal, or the whole in arithmetic progression. The last will be the logarithms, and the former will fill up the logarithmic space.

Cor 4. Let $AZ = 1$ the length of the whole string
 $Zg = l$, any other string.

e = elevation of its tone (above ZA)

E = elevation of tone in an octave.

Then $e = \frac{E \times \log. l}{.30103}$ the elevation of tone when stopt at g .

For ZA , Zb , Zc , Zd , &c. being geometrical
proportionals; and the correspondent elevations of
ound, arithmetical proportionals; the former will
the numbers, and the latter their logarithms.
and since the string $\frac{1}{2}$ is an octave to AZ , and E
elevation of tone; we shall have $\log. \frac{1}{2} (.30103)$
 $E :: \log. : Zg (\log. 1) : e$.

Cor. 5. *If two strings be in a given ratio; or if
be stopt in a given ratio; the sound will be raised
given height.*

For a string double in length to another, or stopt
the middle, the sound will rise an eighth. If $\frac{1}{2}$
stopt, it rises a fifth. If $\frac{1}{3}$ be stopt, it rises a
fourth; if $\frac{1}{4}$ be stopt, it rises a third, &c.

P R O P. IV.

*To investigate the number and distances of all the
notes in an octave, and therefore in the whole scale.* 70.

This is to be done by Prop. II. and its Corolla-
ries, but most easily by the first Corol. By this
Cor. the four strings whose lengths are 1, $\frac{4}{3}$, $\frac{3}{2}$, and
sounding together, give the most agreeable har-
mony. And if they be so delightful when taken in
conjunction, they cannot be otherwise when taken
in succession; therefore we may be assured, that
these four notes actually exist in the scale. Then
our next business is to find at what distances the two
ounds in the middle (given by the strings $\frac{4}{3}$, $\frac{3}{2}$) are
placed above the sound of the lowest (given by 1)
which is the ground; for that given by the string
is fixt, being an octave higher. The four strings
sounding these notes are ba , bg , bd , bc ; then cal-
culating the distances of sound by Cor. 4. of the
st Prop. we shall find that the string ba ,

stopt

Fig. 70. stopt at *g*, raises the sound .322 parts of an octave above *a*.

stopt at *d*, raises the sound .263 parts of an octave above *g*.

stopt at *c*, raises the sound .415 parts of an octave above *d*;

where *d* is an octave to the first at *a*. But these divisions (if they were all equal) would only make 3 notes in an octave, which would be too few to express any music, as there could be no variety therein. Therefore we must subdivide these divisions into less ones, as near equal as we can, so as to be melodious; for that is the ultimate end of our enquiries. If the least $gd = 263$ be taken for the length of a note, *ag* will contain about 1, which does not answer right. If *gd* contains 2 notes, then *dc* will contain $3\frac{1}{2}$, which still does not answer. If *ag* contains 4, then *dc* will contain about 5, and *gd* about 3. This is the nearest division we can get in small numbers. According to which there will be 12 divisions, or 12 notes in the octave almost equal to one another; only these *ag* will be something less than the rest. But here will be too many notes here in the compass of an octave; for if 12 such strings be made to sound in succession, they will give no delight at all; therefore this is not the thing that nature aims at. Therefore we must try to reduce them to a lesser number; and this is not to be done otherwise than by taking half of *ag* for a note; then

we have in *ag* 2

in *gd* $1\frac{1}{2}$

in *dc* $2\frac{1}{2}$

There will then be in all 5 whole spaces and two half ones; but these half spaces cannot be brought into one, to make a whole space, because they happen in different divisions, one being in *gd*, the other in *dc*, which cannot interfere with one another.

her. Here then we shall have the octave divid- Fig.
into 7 parts, 5 of them greater, and nearly 70.
equal to one another; and 2 lesser also nearly
equal, and about half the bigness of the former.
The greater may be properly called *whole notes*;
and the lesser *half notes*. And nature will admit
no more, without the utmost jangling of the
sounds, which cannot be called music.

The next thing is to find the places of these half
notes, so that one be in the division *gd*, and the
other in the division *dc*. And since this is entirely
arbitrary, we must call in nature to our assistance.
Let them then be combined together all the ways
possible by the laws of *Permutation*, so that the
half notes keep their proper divisions. And here
they can only be combined 6 different ways. Then
each of them be tried, by singing or playing
them gradually down from the highest to the lowest,
like the nature of ringing 8 bells; and observe which
music is the most melodious and delightful, and
take that combination for the true order of the
notes in an octave; and that will be found to be
 $1, 1, 1, \frac{1}{2}, 1, 1$; no other combination being
musical. So that here nature and art have jointly
inspired to find out the true system of notes con-
tained in the gamut, on which all music is built.
If the notes and half notes be continued through
several octaves, the notes will stand thus,

$1, 1, 1, 1, \frac{1}{2}, 1, 1, \frac{1}{2}, 1, 1, \frac{1}{2}, 1, 1, \frac{1}{2}, 1, 1, \frac{1}{2}, 1, 1, \frac{1}{2}, \&c.$

where the two half notes are so placed, that there
are always two and three whole notes alternately be-
tween them. And this is an essential property of the
true or gamut. And I affirm, that all tunes, or
pieces of music whatsoever, must be played by a
voice thus constructed; and that nature allows of
no other so perfect and melodious.

Indeed it is almost self-evident, that to have the
series of notes as melodious as possible, that the

B b

half

Fig. half notes be placed as far off one another as the possibly can; so that there must be two and three whole notes alternately placed between every two half notes. For any body may find, by trying upon an instrument, how disagreeable the music is when the half notes are placed together, or separated by one note.

Further, these half notes must keep their place in respect of the lowest note, which note therefore is the *key*. For let as many changes as you will be rung on 8 bells; the music naturally tends to end upon the lowest note; and it is the same thing for any tune. That note then is the *key*, that is, two whole notes above it, and then a half note above and a half note below it.

The half notes give a charming and pleasing variety in all music. If all the notes had been of the same nature equal, there would not have been that variety and sweetness, which the mixture of whole notes and half notes affords.

Now for distinction, names must be given to these 7 notes; and they are denoted by the first seven letters of the alphabet; and that which is the lowest is called C, and the rest in order; which being placed in proper order, and at their due distance to shew the half notes; the gamut will be as follows.

C D E F G A B C D E F G A B C D E F G

So every eighth has the same letters repeated; and the notes, that they represent, are to be written in the same manner for the practice of playing, will be shewn afterwards. If the notes in an octave be reduced to numbers, they will stand thus,

E

G

$$C \frac{8}{9} \times \frac{9}{10} \times \frac{15}{16} \times \frac{8}{9} \times \frac{9}{10} \times \frac{8}{9} \times \frac{15}{16}$$

But $\frac{8}{9}$ and $\frac{9}{10}$ may change places, in the periodical division CE; and also in the division Gc. The fractional numbers denote the parts of the staff which

sound the notes in progression, each part taken from the last length. That is (if ZA be the string, Fig. 72.) $\frac{8}{9}$ of AZ, $\frac{9}{10}$ of bZ, $\frac{11}{12}$ of cZ, &c. are the parts. Or the whole lengths sounding the several notes will be 1, $\frac{8}{9}$, $\frac{4}{3}$, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{8}{9}$, $\frac{1}{2}$; that is, ZA = 1, Zb = $\frac{8}{9}$, Zc = $\frac{4}{3}$, Zd = $\frac{3}{2}$, &c. of ZA, the whole string. And hence $\frac{8}{9}$, $\frac{9}{10}$, $\frac{11}{12}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{12}$, represent the length of the notes, from the key, being equal to Ab, bc, cd, de, &c.

Cor. 1. It is an essential property of the scale of music, that through the whole, there are placed alternately two and three whole notes, between every two half notes.

Cor. 2. It follows from what has been said, that this is the true key of all musical compositions that are lively and merry; or that such pieces of music will all naturally end in that note, and is called the *triple key*.

SCHOL.

It may be observed, that another series of notes might have been taken to make up an octave; and that is, instead of half notes, to join them to the next whole notes; and then we should have 5 notes in the octave; 3 lesser nearly equal, and two greater, being half as much more as the other; and this scale may be said to be the former scale abridged or contracted. And they will stand as follows,

C D E G A C d e g &c.

in numbers $\frac{8}{9}$ $\frac{9}{10}$ $\frac{5}{6}$ $\frac{9}{10}$ $\frac{5}{6}$
where one and two of the proper notes stand alternately between each two of the double notes. It cannot be denied, but that music may be composed from this system; but by reason of too few notes in the octave, the music will not have compass and variety enough; nor will it have that sweetness which it would have by the system of 7 notes; as

B b 2

any

Fig. any one may experience upon an instrument, by striking these 5 notes like ringing 5 bells.

What has been said here, relates to the gay key which ends in C; but there is another key in which all grave and solemn pieces of music always end, which is A, as will be shewn in the next Proposition. But the system of notes is the same for both keys.

Notwithstanding the imperfection of this scale, yet six bells made to strike these notes, would sound very prettily.

P R O P. V.

As the note C in the gamut, is the key in which all merry and joyful music ends; so the note A is the key in which all flat and sorrowful music ends, and to which it tends as to a center.

This is proved by experiments, for the flat key tends in the same manner as was done for the sharp key. For let any tune or piece of music, which excites grief or pity be tried, and it will be found always to tend to A at its conclusion, and to end there. And as this is found always to hold good, then this is naturally the key to all these soft movements. I confess I can give no reason for this, more than I can give a reason why sugar is sweet and wormwood bitter; experience shews both. But I think it is no wonder at all to find that the pieces of music which are sprightly, and those which are doleful should tend to different points, and have different conclusions. When a person in distress is complaining, or lamenting in tears, his voice naturally rises from B to C, and then drops into A. This is the music of a mournful languishing voice, as in the crying of a child. As grief by some mechanism in nature, excites the sort of vocal music; so on the contrary, this kind

Music excites grief, sorrow, melancholy, flatness, Fig. and uneasiness in the mind.

Cor. 1. *Music in the sharp key, or in C, is gay, lively, chearful, sprightly, merry; and inspires joy and gladness. It is known by having a greater third above it; or only two whole notes above, and a half note below.*

Cor. 2. *Music in the flat key A is soft, grave, soft, dull, pensive, sorrowful, melancholy, doleful; and excites grief, pity, woe, sadness, mourning, &c. It is known by having a lesser third above it; or a whole note above it, and 2 whole notes below.*

S C H O L.

These two keys are so different in their nature, that one inspires mirth and the other sadness; and in different degrees. There are certain sounds natural to joy, others to sorrow and despair; others to love and compassion, &c. And a skilful artist can make such compositions, or combinations of notes, as can raise any of the passions of the human soul. Various combinations of sounds produce various effects upon the imagination. Some compositions inspire courage, others terror. Some incline to love, joy, and mirth; others to anger, sorrow, despair, &c. A judicious mixture of consonances and discords have so bewitching and magical effects upon the passions, and so powerfully excite them, and work upon the imagination so strongly, that men frequently run into actions of a correspondent nature, or such as are the natural result of rage, anger, love, despair, &c. So surprizingly do these artful compositions in music raise or depress the spirits.

P R O P. VI.

To determine the number of concords and discords.

It has been observed (Prop. II.) that a concord caused by the frequent coincidence of the vibration of two strings; and a discord by their seldom never coinciding. This being allowed, we may consider how many cases can happen where the lengths of two strings can be expressed by simple numbers; for their ratio denotes some concord.

1. The length of two strings being 1 and 2, denotes an eighth.

2. The lengths being 2 and 3, then $\frac{3}{2}$ expresses a fifth.

3. The lengths being 3 and 4, then $\frac{4}{3}$ denotes a fourth.

4. If the lengths be 4 and 5, then $\frac{5}{4}$ is a greater third.

All these have been shewn in Prop. II.

5. If the lengths be 3 and 5, then $\frac{5}{3}$ is the concord, and (by Cor. 3. Prop. II.) that is equal to $\frac{3}{2} \times \frac{4}{3}$, consequently $\frac{5}{3}$ is a greater sixth, being equal to a fourth, and a greater third; $\frac{5}{3}$ is too little, being less than $\frac{4}{3}$, which denotes an eighth.

6. If the lengths be 5 and 6, then $\frac{6}{5}$ denotes a lesser third (Prop. II.) If the lengths be 4 and 6, then $\frac{6}{4} = \frac{3}{2}$, which we had before. And $\frac{6}{5}$ as before.

7. If the lengths are 6 and 7, or 5 and 7, or 7 and 7; I doubt not but these would make imperfect concords. But they can be of no use in a scale, as they would be incommensurable to the other of the notes.

8. If the lengths be 7 and 8, this concord would also be incommensurable and useless. And $\frac{8}{7}$ is equal to $\frac{3}{2}$, which we had before. But if

lengths be 5 and 8, then $\frac{5}{8}$ will be a concord, be- Fig.

$\frac{5}{8} = \frac{5}{2} \times \frac{5}{4} = \frac{5}{2} \times \frac{3}{4}$, and is equal to a lesser third and a fourth, and therefore is a lesser sixth.

9. If we take 8 and 9, we come among the discords; for $\frac{8}{9}$ is but the distance of a note, and $\frac{9}{16}$ another note, also a discord; and all beyond are discords; till you come at $\frac{1}{16}$, which is still a great discord. Yet notwithstanding these go under the name of discords, they are not properly such; for they produce a sort of harmony, for in the strings 8 and 9, the vibrations coincide at every 72 of one, and every 9th of the other; and that coincidence is the criterion of the degree of harmony. But these that have no such coincidence are properly discords. And hence it comes that these discords $\frac{8}{9}$, $\frac{9}{16}$, &c. mixt with concords, make the most agreeable music.

The number of concords will now appear, to be,

1. The *lesser third* ($\frac{5}{6}$), consisting of a note and half.

2. The *greater third* ($\frac{4}{3}$), containing two notes.

3. The *fourth* ($\frac{3}{2}$), contains two notes and a half.

4. The *fifth* ($\frac{2}{3}$), consists of 3 notes and a half.

5. The *lesser sixth* ($\frac{5}{3}$), contains 4 notes, or 3 notes and two half notes.

6. The *greater sixth* ($\frac{3}{4}$) consists of 4 notes and half.

7. The *eighth* ($\frac{1}{2}$), consists of 6 notes, or 5 notes and 2 half notes.

The number of discords are, the *lesser second* ($\frac{15}{16}$), the *greater second* ($\frac{8}{9}$ or $\frac{9}{10}$), the *lesser seventh* ($\frac{10}{18} = \frac{5}{9}$), the *greater seventh* ($\frac{16}{30} = \frac{8}{15}$).

The greater fourth is a discord when it consists of 7 notes, and is the same as a flat fifth, as from B to F. So that the number of concords is 7; and

Fig. the number of discords 5. If we go without the compass of an eighth, it comes to the same thing for an eighth, ninth, tenth, &c. are the same as the unison, second, third, &c. The 8th and 5th are called perfect concords; the rest imperfect.

Cor. 1. *The lesser sixth is equal to a lesser third and a fourth; a greater sixth is equal to a greater third and a fourth. An eighth is equal to a fifth and a fourth, and to a lesser third and greater sixth, and to a greater third and lesser sixth, and to a second and seventh; which may be called complements to one another.*

Cor. 2. *Of concords, the 8th is most perfect, the 5th, and next the thirds, and then the sixths.*

Cor. 3. *Of discords, the lesser seventh and greater second are the sweetest, being complements. And the greater seventh and less second the most harsh, being also complements. The fourth (greater) is also harsh.*

P R O P. VII.

To find the divisions of a monocord to sound the concords, according to the number of notes contained in each; upon supposition, that all the notes in the gamut rise by equal degrees; and to shew the difference between these and the harmonic divisions; or between the isothetic and harmonic scale.

This Problem is solved at large in Prob. XXI *Algebra*, and in Ex. 108. of the *Mechan.* 4to. The particulars of which I shall here put down. There are 6 notes contained in an octave, or 12 half notes. But these notes are not all of a bigness in all the concords, as is shewn in the *Mechanics*; so the notes do not rise therein by equal intervals of sound, according to that harmonic division. Let

ZA be the string $= 1$, $Zb = \frac{1}{2}$. And let $Zb, Zc, Fig. Zd, \&c.$ sound the half notes, and put $Zb = x$, 71. then Zb is the first of 12 geometrical proportionals;

whence is found $x = \sqrt[12]{\frac{1}{2}} = .9439$ for half a note; and $Zc = xx = .8909$ for a note. And $x^3 = 8409 = 1\frac{1}{2}$ notes, $x^4 = .7937 = 2$ notes, $x^5 = .74915 = 2\frac{1}{2}$ notes; $x^7 = .66742 = 3\frac{1}{2}$ notes, $x^8 = .62996 = 4$ notes, $x^9 = .5946 = 4\frac{1}{2}$ notes, $x^{12} = .5 = 6$ notes. These are the divisions for each of the concords in order, for ascending equally.

But the harmonic divisions will be $.8\frac{1}{3}, .8, .75, .625, .6, \frac{1}{2}$. These put into decimals, and set down in a Table with the former, will be as follows.

names of the cords.	pure concords.	equal ascents.	errors.
wh. string	1.00000	1.00000	0
lesser third	.83333	.84090	$\flat \frac{1}{9}$
greater third	.80000	.79370	$\sharp \frac{1}{13}$
fourth	.75000	.74915	$\sharp \frac{1}{100}$
fifth	.66666	.66742	$\flat \frac{1}{100}$
lesser sixth	.62500	.62996	$\flat \frac{1}{13}$
greater sixth	.60000	.59460	$\sharp \frac{1}{13}$
eighth	.50000	.50000	0

Then to find the errors or variations of the correspondent cords. Among the methods made use of in the Algebra, we have this (which is also in the Mechanics); Ac or the length of a note at A is $= 1 - xx = .1091$; and the length of a note at any other place as d , that is $de = Zd \times .1091 = P$. And let Zd represent any of the numbers in the third column, and s that in the second, corresponding. Then the error will be $\frac{Zd - s}{P}$, for that concord. As in the fourth, $Zd = .74915$,
 $s = .75$

Fig. $s = .75000$, and $P = .74915 \times .1091 = .0817$;

7th. and $Zd - s = .0008$, then $\frac{Zd - s}{P} = \frac{.0008}{.0817} =$

$\frac{1}{100}$ nearly, the error in the fourth. The errors in all the concords being thus computed, are set down in the Table, col. 4th. This last column shews the quantity of the error, or what part of a note it is; and the mark $\sharp$ shews the note (by equal ascent) to be too sharp, and the mark $\flat$ shews when it is too flat. Here these concords that are the complements to one another have the same error, but of different denominations.

Cor. 1. Hence, in this isotonical scale, since an eighth contains 6 whole tones; then computing from the number of notes in each concord, as found in the last Prop. we shall find 4 lesser thirds = an eighth, 3 greater thirds = an eighth, 12 fourths = 5 octaves, 12 fifths = 7 octaves, 3 lesser sixths = 2 octaves, and 4 greater sixths = 3 octaves.

Cor. 2. The isotonical scale, as here described, is the best scale for practice, that can be contrived.

For in the scale described in Prop. IV. which is the natural scale, and the only one that nature has laid down; the notes are not all equal, some are denoted by $\frac{9}{8}$ and some by $\frac{5}{4}$, the difference is $\frac{1}{80}$ or a tenth part of one of them; which makes no sensible difference in playing upon a single instrument. But in composition, it does not answer; for if a concord can be taken perfect from some of the notes, it will be imperfect when taken from others, which occasions great confusion; but the isotonical scale answers equally for all; and the difference from the true scale is insensible. For the greatest error is in the lesser third and greater sixth, which amounts but to the thirteenth part of a note, which makes an imperceptible variation from the true cord,

cord, when played in consort; and which no ear can discover. For such an error is not so discoverable in a third or sixth, as in a fifth or eighth; for the sweeter the cord, the sooner an imperfection is discovered. But the error in a fifth must be quite insensible, being but the hundred part of a note. And although these trifling errors will a little diminish the sweetness of the harmony, yet there is no remedy, but what is worse than the disease.

In the natural scale, the difference of a fourth and fifth is $\left(\frac{3}{4} - \frac{2}{3}\right) \frac{1}{12}$, for the string $\frac{3}{4}$; or $\frac{1}{12}$ for the string 1; therefore $\frac{8}{9}$ will be a note, and therefore $\frac{8}{9}$ should be $= \frac{1}{12}$, but it is only .4933. For there are 6 notes in the octave.

Again, take the difference between the fourth and lesser third, and we shall have $\frac{1}{10}$ for a tone. But $\frac{9}{10} = .5314$ instead of $\frac{1}{10}$. So that neither $\frac{8}{9}$ nor $\frac{9}{10}$ is an exact tone.

A lesser third, a greater third, and a fourth, differ by half a tone, and there should be 12 in the octave. In the first we have $\left(\frac{5}{6} - \frac{4}{5}\right) \frac{1}{10}$ for the string $\frac{5}{6}$ or $\frac{1}{10}$ for the string 1, and the string half a note above is $\frac{2}{3}$, but $\frac{24}{25} = .6127$, which

should be $\frac{1}{10}$. And in the second $\left(\frac{4}{5} - \frac{3}{4}\right) \frac{1}{20}$ for the string $\frac{4}{5}$, or $\frac{1}{20}$ for the string 1. And $\frac{15}{16} = .4609$, which should be $\frac{1}{20}$. Therefore the first is too little, and the last too big, for a half note.

As none of these notes or half notes will make up an octave, so neither will any number of thirds, fourths,

Fig. fourths, or fifths, make one or more octaves. For

then (by Cor. 1.) we should have $\frac{5}{6}^{14}$, or $\frac{4}{5}^{13}$, or

$\frac{3}{4}^{13}$, or $\frac{2}{3}^{12}$ = $\frac{1}{2}$; but none of these are so;

and therefore we may conclude, that no scale made up of such notes or half notes, or any combinations of them, or of perfect thirds, fourths, fifths, &c. can be perfectly exact. And therefore the isotonic scale, whose numbers are in the third column of the Table, must be the best and most practical system.

They that from the division of the monocord insist, that all instruments should be turned according to that division, which is $\frac{8}{9}$, $\frac{9}{10}$, $\frac{15}{16}$, $\frac{8}{9}$,

$\frac{9}{10}$, $\frac{8}{9}$, $\frac{15}{16}$, don't consider, that thereby they bring

the practice of music into a very narrow compass. For it must always be performed only upon one key. For the notes can never have the same position, but in that situation of the key.

Cor. 3. *Therefore all instruments ought to be tuned according to the isotonic scale. And in the harpsichord, &c. when tuned by fifths, the upper note must be taken as flat as the ear will bear, and also lesser thirds; but greater thirds must be taken as sharp as they will bear.*

It is the practice of most people in tuning, to take the fifths perfect, till the last, which they leave imperfect; and such imperfect notes they call *bearing notes*. But this spoils the music, for a fifth taken on that note is no better than a discord, and therefore ought to be avoided in composition, which will cramp the music, and perhaps quite spoil the air of the piece. For there will be 12 removes

or

of repetitions before one comes to the same note Fig. again; and since in one fifth there is $\frac{1}{1080}$ error, in 12 fifths there will be $\frac{12}{1080}$ or $\frac{1}{90}$ of a note at the last, which will be very sensible in a fifth; and the errors will be far greater in a third or a sixth. Therefore those that chuse to have some of the concords perfect, must be forced to make others as imperfect, and often so bad, as to be no better than discords.

Another disadvantage would be, that the music could not be transposed upon any other key, as occasion required; but tied down to a very few; and upon these the same inconveniencies would happen.

Some people, to cure the imperfection of the scale, have invented quarter notes, which makes the music hard to play, and does not take away the disease neither. Therefore all things considered, I am persuaded that the isotonic scale is the best for practice, as it is attended with no inconveniencies at all. The errors arising by using that scale, does not amount to so much as the alteration of the weather, by heat or cold, drought or moisture, causes in a little time, either upon strings or pipes. Therefore the best way to remedy these errors, is to divide the defect equally among the notes of the instrument; so that what might be sensible in one note, may escape the senses when divided among many.

The proper way to tune then is this; begin at the middle of the instrument, and first tune the fifth perfect, then lower the upper note a small matter; do the same with all the rest till the octave be finished. Then if the last note be either too high or too low, begin anew, and accordingly alter all the notes a little, till the last does agree; but this art is gained by practice. This octave ought to be tuned exactly, for the rest depend upon that; and there

Fig. there is no more to do, but to make them perfect octaves to these, both above and below.

Cor. 4. If A, B, C, D, E, F, be the length of the strings, to sound the lesser third, greater third, fourth, fifth, lesser sixth, greater sixth, respectively;

$$\text{then } A = \frac{1}{2} |^{\frac{1}{4}}, B = \frac{1}{2} |^{\frac{1}{3}}, C = \frac{1}{2} |^{\frac{2}{3}}, D = \frac{1}{2} |^{\frac{7}{12}}, E = \frac{1}{2} |^{\frac{2}{3}}, F = \frac{1}{2} |^{\frac{1}{2}}.$$

For these are respectively $= x^3, x^4, x^5, x^7, x^8, x^9$.

SCHOL.

The gamut or system of notes is the alphabet of music. For as the alphabet contains all the letters used in writing; so the gamut contains all the notes used in music.

PROP. VIII.

To enlarge the scale, and make it more comprehensive; and to shew the use of it.

This is done by dividing all the whole notes into half notes; so that instead of 7 notes, in the natural scale, there will be now 12 notes in this artificial scale.

By the last Prop. we found $x = \sqrt[12]{\frac{1}{2}} = .94390$ which is the first of the half notes; then we must find $x^2, x^3, \&c.$ to x^{11} , for the rest of the half notes; and these will be as in the following Table.

notes.	isotonic scale.
ground	1.00000
$\flat$ second	.94387
$\sharp$ second	.89090
lesser third	.84090
greater third	.79370
fourth	.74915
$\sharp$ fourth	.70711
fifth	.66742
lesser sixth	.62996
greater sixth	.59460
$\flat$ seventh	.56123
$\sharp$ seventh	.52973
eighth	.50000

The scale will now stand thus,

A. A $\sharp$. B. C. C $\sharp$. D. D $\sharp$. E. F. F $\sharp$. G. G $\sharp$. A

or A. B $\flat$. B. C. D $\flat$. D. E $\flat$. E. F. G $\flat$. G. A $\flat$. A

These repeated make up the whole scale, or universal gamut, by which all music that ever was, or can be, in the world must be pricked or written.

But notwithstanding this pompous shew of notes, no more of them can be made use of in playing, than was before used in the natural scale of 7 notes, laid down in the 4th Proposition. It may be asked then, what use these additional notes are of, and for what purpose are they put in? The use of them is to fill up the vacancies of the common scale, and to make it uniform, so that any piece of music may be pricked as high or as low upon it as one will.

A tune in the sharp key can only be pricked upon the natural scale, to end on C; because the two half notes must be confined to their places, so far above or below C. But in this scale there are half notes every where; so that the tune may be set to end upon any key, and the situation of the half notes

Fig. notes remain as before. But then the rest of the half notes are not to be played, and are in that case useless; for in every case there are only 7 notes to be played in every octave, the rest having nothing to do in the music. For suppose there is a tune that ends in C, or has its key in C; then all the flats and sharps, you see there are useless, and must be omitted. But suppose that tune to have its key in D, then you must have C $\sharp$ the next note below, and C must be left out. Likewise you have F $\sharp$, and F must be left out; and moreover all the other flats and sharps that you see must be left out, for they have no business. And thus the proper situation of the two half notes is preserved the same as before, when the key was in C.

In modulation, or passing from one key to another, nothing can be done without proper flats or sharps. For by the judicious use of some of these the musician passes almost imperceptibly from one key to another; and then he uses such flats or sharps as are proper for that key, and all the rest must be let alone; and also the proper notes belonging to such flats and sharps as he uses, must be omitted. This modulation, or passing from one key to another in playing, is the same thing as transposition, or removing a tune from one key to another in writing or pricking. The rules whereof I shall shew in the next Prop.

P R O P. IX.

To shew the nature, reason, and practice of transposition.

Transposition is the writing any piece of music higher or lower upon the scale, so as to answer the same purpose in playing. The art of transposing consists

consists in setting such a number of flats or sharps Fig. as shall keep the two half notes at such a height and distance above the new key, as they had in the former. And the flats or sharps must be so placed, and no otherwise, that there may be always two and three whole notes alternately between every two half notes, which is to be done after the manner I am now going to shew. The notes in the natural scale stand thus (*n. 1*); and where the half notes are, the letters are set close together.

1. A BC D EF G A BC D EF G &c.

2. A BC D E FG A BC D E FG

Then it is impossible I can observe that law any otherwise, than either by putting F up towards G with a sharp, as in *n. 2*. Or else bringing B down towards A with a flat, *n. 3*.

1. A BC D EF G A BC D EF G &c.

2. AB C D EF G AB C D EF G

In the first case (*n. 2.*) we have 3 notes, and 2 alternately, which answers one property; then we shall find the sharp key to be in G, because there is a half note below it and two whole notes above it, which answers another property. See Prop. IV. Cor. 1. and Prop. V. Cor. 1. and 2.

In the second case (*n. 3.*) we have also 2 and 3 notes alternately; and the sharp key will now be in G, because there is a half note below it, and two whole notes above it. And it is impossible in either case, that the key can have any other situation.

What is here said of the sharp key, is equally true of the flat key, because it is always a note and a half below the sharp key. For if A be the flat key (*n. 1.*) then E will be the flat key (*n. 2.*); for according to its property, it has a whole note above it, and two whole notes below it. And (*n. 3.*) where B is flat, the key will be in D; for there it has one whole note above it, and two below it.

Fig. This foundation being laid, I shall proceed to shew where the key falls for one, two, three, four, &c. flats or sharps, being given. And on the contrary, the key being given; to find what flats or sharps it must have: *and first for the sharps.*

1. A BC D EF G A BC D EF G &c.
2. A BC D E FG A BC D E FG &c.
3. A B CD E FG A B CD E FG &c.
4. A B CD E F GA B CD E F G &c.
5. A B C DE F GA B C DE F G &c.
6. AB C DE F G AB C DE F G &c.
7. AB C D EF G AB C D EF G &c.

And so on till all the notes be sharped, but the practice never goes so far; but it serves to shew how nature directs the whole process.

N. 1. is the natural scale for either key, no matter which; they are now in A and C. All the rest follow from one another, thus;

N. 2. F must be sharp, and the keys fall in B and G.

N. 3. C must be sharp, and the keys are in B and D.

N. 4. G must be sharp, and the keys are in F and A.

N. 5. D must be sharp, and the keys fall in C and E.

N. 6. A must be sharp, and the keys are in G and B.

N. 7. E must be sharp, and the keys are in D and F#.

Here it is evident the notes must be sharped in this order only, F, C, G, D, A, E, &c. *Again for the flats.*

1. A BC D EF G A BC D EF G &c.
2. AB C D EF G AB C D EF G &c.
3. AB C DE F G AB C DE F G &c.
4. A B C DE F GA B C DE F G &c.
5. A B CD E F GA B CD E F G &c.
6. A B CD E FG A B CD E FG &c.
7. A BC D E FG A BC D E FG &c.

N. 1.

N. 1. is the natural scale, the keys being in A Fig. and C; and each number follows from the foregoing one.

N. 2. B must be flat, and the keys fall in D and F.

N. 3. E must be flat, and the keys fall in G and B $\flat$.

N. 4. A must be flat, and the keys are in C and E $\flat$.

N. 5. D must be flat, and the keys are in F and A $\flat$.

N. 6. G must be flat, and the keys are in B $\flat$ and D $\flat$.

N. 7. C must be flat, and the keys fall in E $\flat$ and G $\flat$.

Here it is plain the notes must be flatted in the order B, E, A, D, G, C, &c. and in no other.

It has now been shewn where the key falls, for every number of flats and sharps. Now I shall shew how to find what flats or sharps belong to a tune, when the key is given; or which is the same thing, when a tune is to be transposed so many notes higher or lower as one shall choose. And this is to be done by the reverse method, and is best shewn by examples.

Suppose I have a tune with two flats, and I want to transpose it to have no flats. Look in the last catalogue for two flats, and the flat key will be in A, but where there are no flats, the same key is G, which is a note higher. Therefore the tune must be set a note higher. But suppose you want to set it so many notes higher or lower, as suppose two notes lower. Look for two flats in the latter catalogue, and you will find the key in G; then have it two notes lower, the same key must be in E, therefore looking for E, you will find it with two flats; but this will not answer the purpose; therefore look for E (the flat key) among the sharps (in the first catalogue); and you will find it has one sharp, which must be in F; so that the tune may be set two notes lower by putting F sharp.

Fig. Again, if a tune has 3 sharps, and I want it with only one. With 3 sharps, the sharp key will be in A; and with one sharp in G; so the tune must be set a note lower. But if I want it two notes lower, then the same key must be in F, which requires 6 sharps, which will do nothing; therefore we must take one flat, to have it in F.

73. But the general scale, derived from the foregoing process, will be more expeditious. By this any tune may be set as many notes higher or lower as we please; or to have as many flats or sharps as we will, not exceeding four.

To set a tune higher or lower. Seek the flats or sharps given in your tune, and observe the note of the same bar. Then find the note that is so much higher or lower; and the flats or sharps that are joined with it, must be set to the tune.

To set a tune to have any number of flats or sharps. Find the flats or sharps given in the tune, and then the flats or sharps to be set to it. Then the two correspondent notes will shew how much higher or lower the tune is to be pricked.

For example. Suppose a tune that has 2 flats, to be set 2 notes lower. Here G is the note along with 2 flats, therefore E (which is two notes lower) is the note sought, with which there is one sharp, which answers the question.

Again, if a tune with 3 sharps is to be transposed to have but one. Find 3 sharps, which is at A, then find one sharp, which is at E, and that is one note lower than F. Therefore the tune is to be pricked one note lower. But if I want it 2 notes lower, find D, which gives one flat.

Many Corols. may be drawn from this Proposition. I shall mention some.

Cor. 1. Hence any tune may be transposed, so as to have either flats or sharps upon it. And therefore,

Cor. 2. *A tune is not said to be on a flat or sharp key, because of the flats or sharps set before it. But it is said to be in a flat key, when it has the lesser third above it; and in a sharp key when it has the greater third.* Fig. 73.

Cor. 3. *If a tune requires a single sharp, it can be no way but in F. If it requires two, they must be in F and C. If three, in F, C, and G, &c. and no way else.*

Cor. 4. *If a tune requires one flat, it must necessarily be in B. If it requires two, they must be in B and E. If three, in B, E, and A, &c. and no where else.*

Cor. 5. *In modulation, or passing from one key to another; the musician must play such flats and sharps as are proper to this last key, while he continues in it; omitting the proper notes.*

For playing in another key, is just the same thing, as when a person has been singing a-while; and then another person with a voice so much smaller, should begin to sing some division of the same tune. It is evident he must keep the same air, i. e. observe the same flats or sharps.

Cor. 6. *Let a tune be transposed how it may, still there can be no more notes played but seven in an octave.*

For in the natural scale there are but 7 notes; and there are no more when transposed; though some notes are changed for others, flat or sharp.

P R O P. X.

To explain the gamut, or the method of writing down notes for the practice of music.

I have already demonstrated the natural constitution of the scale of musical notes, how many

Fig. notes and half notes are in an octave, and in what order they are placed. I am now to shew how they must be written down, so as every written note may be distinctly known, and may fitly represent the several sounds they stand for; so that the practitioner may readily play them when he sees them upon paper, which is the design and perfection of this art.

74. The gamut for any single part of music is made by writing the notes upon a staff of five lines, formerly they used six lines; and giving names to every line and every space; and upon these lines and spaces all the different notes are written in order. But to take in the whole compass of music. The ancients invented a gamut consisting of eleven lines, as PQ, and as many additional lines, as they had occasion for, above or below; and upon this staff of lines, they writ all the notes that can be used in music, as *g, a, b, c, d, &c.* The first or highest staff of five lines RS, they allotted to the treble; the lowest five lines TV to the bass, and the middle line, and two on each side, to the tenor: and upon each part, they placed the cleffs to distinguish them; and placed the notes on all the lines and spaces, with their names. The cleffs, and the names placed where the notes are to stand, are set down in the figure. And this scale is used to this day.

But certainly the inventors were very unlucky in forming such a gamut, and the moderns are more so, for continuing the use of it. There is no manner of advantage by having it in this form, for no body ever played a single part, off a scale of eleven lines. For music is always divided into parts, and each part is separately pricked upon five lines. But the disadvantage of this scale is enormous, and is the worst scheme that any body could have fallen into. For taking all the parts of

corresponding to the several parts of music, and Fig. there are no two parts of it alike. For the treble 74- five lines has the highest line F; the base five lines has the highest line A; the tenor five lines has the highest line G, and sometimes B and E, according as the tenor cliff is placed, which by this contrivance must always be C. Nay, it is sometimes 75. placed upon the highest and lowest lines. The consequence of all this is, that whoever would learn to play upon the harpsichord or spinet, must be obliged to learn 5, 6, or 7, different gamuts, when one might have done the business. But who can ever have the patience, or spare the time to do this; certainly none but such as have nothing else to do, or make it their business. Now had but the inventors made their scale of 12 lines instead of 7, and had taken the lowest five for the base; it would have been uniform with the treble; and then there would only have been one gamut to learn, instead of two, now in use, for these two parts, the treble and base, which are the principal parts. But they had not even the good fortune to do this. Would not any body wonder at the patience of a person, that for variety, should make a new alphabet, by making *b* sound as *a*, *c* as *b*, *d* as *c*, &c. and so learn to write and read that way; any person acquainted with the present alphabet, would hardly think that the word *pass* was designed to express *one*. This scheme of five or six different gamuts is of a similar nature. But I am not the first that has taken notice of this absurdity.

It is certain, that the authors of this confused and intricate scheme had never consulted nature, which in all physical enquiries ought to be our chief guide; if they had, she could have pointed out a plain and direct method, which would have prevented the infinite perplexity which a learner is thrown into, by so many inconsistent and contradictory

Fig. dictory methods of notation. For can there be

75. greater absurdity, than to make the same line of space, sometimes to stand for one note or letter and sometimes for another, even through the whole alphabet of notes. And I have often wondered that they did not even use different letters for the correspondent notes in different octaves. And this shews that music has not met with the same improvements, that other arts and sciences have done.

I shall now describe that method of notation which nature has laid down, and which prevents all that nonsensical inconsistency, confusion, and puzzling, which has been so justly complained of. It is acknowledged by all persons that know any thing of music, that all octaves are the same, and have all the same properties, and the same similitude throughout; all octaves then, being so near akin, ought to be so pricked down or written, that the schemes or draughts may have the same similitude or agreement, as their originals. Therefore

76. let the highest octave, or the treble part be set upon a stave of five lines, with the highest line F, as we have it now. Let the second octave or tenor be also set upon another stave of five lines, and let the highest line be F, as before, but an eighth below the first. Lastly, let the lowest octave or bass be set upon another stave of five lines, and the highest line F, which is two eighths below the first. Then let the proper cliffs be set upon each; the treble cliff upon G, the tenor cliff upon C, the base cliff upon F. So there will be but one gamut or one order of notes, for all three; each an eighth below the other. I make no alterations with the treble, but make it the same as before, because it is universally known; and therefore I accommodate the rest to that, which are but known to few. I design each of these to be further extended than five lines, by drawing occasionally other lines either

either above or below, as is practised now. This Fig. 78. method is agreeable to the nature of things, consistent with itself, and the easiest that can be thought of. And I will venture to say, that whoever has a desire to play upon the harpsichord, will learn in a quarter of the time, that the common way requires. And that many people are deterred from learning, from the apparent difficulties in the common formidable scheme.

The only objection I can think of, is this; all our music is written according to the vulgar method; and in that form can be of no use. But that is easily remedied, by transposing it, as occasion requires; as I have often done myself; which will not be a great deal of trouble; and will, besides, teach the scholar to write music.

I do not mean by this, that any person who has had pains and patience enough to learn the common way, should lay it aside to take this; for he need not have two methods, to do one thing. Yet those persons, if they know the common way never so well, would fall into this method, without the least trouble or difficulty. And I believe if this method was once begun, it would soon become universal. You have this gamut in Fig. 82. where A, B, C, D, &c. are set to the notes belonging to them, and whose places are invariable. I have not put the names at length, A lamire, B fabemi, C solfant, &c. because the letters A, B, C, &c. serve my turn as well.

There is a like absurd practice made use of in singing; which is, that when B is flat, then Mi (whose true place is B), and the whole train of notes attending it, are to change their places; and Mi is to fall in E. Then if a flat happens in E, they are all to change their places again, and Mi must fall in A, and so on for more flats. As if Mi, or any other syllable, could not be applied to any

Fig. any sound, or musical note, flat or sharp. A method so absurd and irrational, that no reason can be given for such a practice, but the humour of making every thing as obscure and difficult as it can possibly be made. And some do not scruple to say that music has been designedly involved in all this darkness and obscurity by the professors of this art in order to keep the secret to themselves; for it is certain they go the furthest way about. As to singing, there can be no occasion for any names for the seven notes, but the seven letters A, B, C, D, E, F, whose places are eternally fixt. What then is all this pother about? Such are the inconsistencies men run into, when they forsake the plain path of nature, to follow precarious methods, that have no foundation in the nature of things.

P R O P. XI.

To describe the several sorts of notes used in music, and other marks and characters used therein and of time.

79. Notes are of several forms, according to the lengths of the sounds they are to express. A a *brief*, B a *semi-brief*, C a *minnim*, D a *crotchet*, E *quaver*, F *semi-quavers*, G *demi-semi-quavers*. Their length in time of playing is, a brief (now out of use) two semi-briefs. A semi-brief, two minims. A minim, two crotchets. A crotchet, two quavers. A quaver, two semi-quavers. A semi-quaver, two demi-semi-quavers; their tails are often tied together, for the benefit of the eye. A *prick* set after a note makes it half as long again. The heads of the notes must stand in the proper places; the tails may be turned any way for convenience.
80. *Rests* are marks to shew, that the music stops for a certain length of time, expressed by the length

of some note. The rests are set along with the Fig. notes of the same length.

A *flat* (A) is a mark set before a note, to shew 81. it is to be played half a note lower.

A *sharp* (B) shews, that the note before which it is set, is to be played half a note higher.

A *natural* (C) set before a note, shews the note must be played without flat or sharp.

When any flats or sharps are set at the beginning of a lesson, every note upon such lines and spaces must accordingly be played flat or sharp throughout; and all their octaves.

A *repeat* (A) shews, that the tune is to be played 83. over again from that mark to the end of the strain, or to the *close* E.

A *direct* (B) placed at the end of a stave, shews what note begins the next stave.

A *shake* (C) is performed by shaking the note above.

A *beat* (D) is performed by shaking the note below.

A *stave*, or *staff*, is the five lines drawn to write the music on, as in Fig. 76, 77, &c. The additional lines above or below are called *ledger-lines*.

Bars, are lines drawn across the stave; a single bar is a single line; and these go quite through the tune, and are to divide the time equally through the whole part. For every bar (contained between one cross line and another) is of the same length, and must consist of the same number of any one sort of notes. The double bars have two cross lines, and are only set at the end of a strain, or part of the music.

Time, is that kind of movement which is proper to the tune, and is of two kinds, common time and tripple time.

Common time, is when the notes move by two and two. In a tune, or lesson in common time, a bar

Fig. bar contains the length of some particular note, a semi-brief, minum, &c. which may be divided

84. at pleasure. There are several moods in common time, denoting a swifter or a slower motion. The slowest is marked as you see at A; the next marked at B is faster; third at C is the quickest; sometimes they are marked as at D and E, and are of the same measure as C, and have but two crotchets in a bar, while the rest have four. The length of a bar in the swiftest (C) is a second; and in the slowest, two seconds.

Tripple time; here the motion is by three and three; and there are always three in a bar of some sort of notes; or twice three, or some number of three's. The moods are as follow.

The slowest kind is marked thus $\frac{3}{2}$, and has minims in a bar, the next is 3 or $\frac{3}{4}$; here is a pricked minim in a bar.

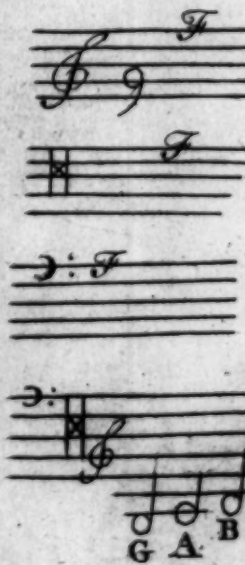
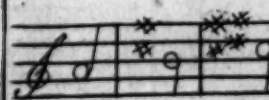
The quickest is marked $\frac{3}{8}$; here is a pricked crotchet in a bar. There are also some compound moods, as $\frac{6}{4}$ and $\frac{6}{8}$; here the crotchets or quavers go by 3 doubled, that is, two three's in a bar.

Also in the moods $\frac{9}{4}$ and $\frac{9}{8}$, the crotchets or quavers go by 3 trebled, that is, by three three's in a bar.

Lastly, there is the mood $\frac{12}{8}$, in which the quavers go by 3 quadrupled, that is, by four three's in a bar.

A general rule to know the movement belonging to any of these moods, or how it is barred. Let a semi-brief be the integer, and then the fraction denoting the mood, will tell you what part or parts of it are in one bar. Or which is the same thing take the part of a semi-brief denoted by the lower number; and the upper number will tell you how many of these parts are in a bar. Thus in $\frac{9}{4}$, the 4th part of a semi-brief is a crotchet; therefore there are 9 crotchets in a bar.

Th



Miscellanies

H
 F
 C

A ⁶⁹ B A b c d e f g h 71 Z
 a h g f d c b
 70 A b c d e f g h 72 Z
 a g d c b

73

74

R
S
T
Y
V

75

C C C

79 A B C

76

77

78

D
 E
 F
 G

80

81 A b . B * . C 4

82

A B C D E F G A B C D E F G A B C D

83. A : S : B w . C ^{tr} . D x . E ∆

The distinguishing mark of common time is, Fig. that all notes go by two's; and that of tripple time, that some particular note goes by three's, whilst all the rest go by two's. And in all sorts of time, the movement is compleated at the end of every bar; at which, there is a kind of intermission or stop, which divides the tune regularly into similar parts or elements. And it is probable, that the compound moods in triple time have been contrived to answer some sorts of measures in poetry, or verse; as we have songs in many different sorts of measures, or number of syllables in a verse. And these moods have been fitted to answer them.

There are several other marks used in music, but these are the principal.

P R O P. XII.

To explain the phenomena of communicating sounds from one string to another.

1. If two strings of any instrument be tuned unison, and one of them struck, and made to sound; the other will give a sound also. Or if the strings be upon different instruments, the effect will be the same, if they are placed near together.

The reason of this is, that if the string AB be struck, and made to vibrate, the other string, though at rest at first, will receive all these impulses of the air, which must necessarily put it into motion; for every force acting upon a body at liberty, will excite a certain degree of motion; and this must be the same vibrating motion which the air has; and that is the same it receives from the vibrating string. And having the same vibrating motion, it must give the same sound. This is the case with the drum of the ear, when it receives the impression from the tremulous air, and causes the sound

Fig. sound to be heard. For as the vibrating motion

85. of a string communicates a like motion to the air; so, *vice versa*, the vibrating motion of the air communicates a motion to a string, that is, in a proper disposition to receive it.

86. 2. If two strings be tuned to an octave, and the higher string be struck, the lower will give the same sound; as may be tried on a fiddle. If AB be an octave above AC, and AB struck, then AC will give the same sound. But then it is not the whole of the string AC that trembles, but the two equal parts AD, DC, which are in unison with the first AB. The reason here is the same, for the impulses of air from the string AB, cannot give the same motion to the whole string AC, because it can naturally vibrate only half as fast, and therefore one impulse destroys the effect of another, for producing this motion. Therefore the effect of these several impulses will at last be this, that the string will be forced into two vibrations of the two equal parts AD, DC; which are in unison with AB. For the middle point is at rest, for a small piece of paper laid on D, will be unmoved; but shifting it gradually towards either end, it will be thrown off, or made to tremble.

87. Likewise if the string EF be a twelfth below AB, or a fifth below AC, and EF be divided into 3 equal parts, these three parts will vibrate, whilst the points of division are at rest. For the vibrations of the air, acting on the string, will move the parts separately, being each in unison with AB. And if EF was two eighths below AB, then the four parts of EF would move; but the motion at last grows so languid and faint, that it cannot be perceived.

But if any of these strings are struck at the points of division, it will only give a confused sound; for the motions are then inconsistent, by reason the

point

points are disturbed that should be at rest. A pipe Fig. will also cause the like vibrations and sound in a string, that is unison with it.

3. If a string be tuned a perfect twelfth to the base, and another string a 17th to it, or a greater ratio to the other note. And if the base be struck, the two others will sound; but it is only a good ear can perceive them. I take it for granted, that the impulses of the air acting equally on both sides of a string, will give it no motion; but acting more on one side than the other, will give it a motion proportional to that excess. Here in the first case, the lengths of the strings are 1 and $\frac{1}{3}$, therefore the upper string vibrates 3 times for the lower once. Divide the time into 3 equal parts, each equal to the vibration of the upper string; and let the vibrations of each, towards the right and left, be denoted by *r* and *l*, and their conspiring or opposing one another, that is, the addition or diminution of motion of the upper string, by *a* and *d*. Then their motions will stand thus,

lower	<i>rrr</i>	<i>lll</i>	<i>rrr</i>	<i>lll</i>
upper	<i>rlr</i>	<i>lrl</i>	<i>rlr</i>	<i>lrl</i>
	<i>ada</i>	<i>ada</i>	<i>ada</i>	<i>ada</i>

In this comparison, it appears there are 2 *a*'s and 1 *d*; and consequently $\frac{1}{3}$ of the vibrations concur to move the upper string, the remaining two destroy one another. Therefore the upper string will be made to vibrate, by the vibrations of the lower.

The other string (the 17th) is $\frac{1}{5}$ in length, which will give 5 vibrations for one of the base. Then proceeding down as before, which way these vibrations (parts of vibrations) lie; they will stand thus,

lower	<i>rrrrr</i>	<i>lllll</i>	<i>rrrrr</i>	<i>lllll</i>
upper	<i>rlrlr</i>	<i>lrlrl</i>	<i>rlrlr</i>	<i>lrlrl</i>
	<i>adada</i>	<i>adada</i>	<i>adada</i>	<i>adada</i>

By

Fig. By this it appears, there are 3 *a*'s for 2 *d*'s, therefore $\frac{3}{2}$ part is spent in giving motion to the upper string; for it is certain, where the motions conspire (the *a*'s), it will be accelerated, and where they oppose (the *d*'s), it will be retarded. And in the whole, it will be made to vibrate. The string $\frac{3}{2}$ will also be made to vibrate a little; but all of them give so small a sound, as only a good ear can perceive. But none besides, as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$ &c. can be made to vibrate upon this account, from the vibrations of the base; as may be easily computed after the same manner.

There are many more such phænomena of musical sounds, which my room will not admit of. I shall only observe, that some sounds excite others as has been shewn, and these again create others and these produce more; and all by setting such strings a vibrating as are in a disposition to receive the motion, that is, whose vibrations any way fall in with the others. So that at last there is a strange medley of musical sounds, chiming one with another, as may be observed in any full stringed instrument of music.

P R O P. XIII.

To lay down the general rules of taking concords and discords in composition.

It has been shewn before what notes sounding together are concords, and what are discords; we must now shew how they must be put together as to make the best harmony. For if they be put together at random, they will but make indifferent sort of music; therefore musicians have, by trial and experiments, found out and laid down rules how they are to be managed, to make the music as sweet as possible. In the first place a base is to

be framed, which must express the air of the tune, Fig. and must move, not so much by degrees, as by leaps of a 3d, 4th, or 5th. It must have proper closes, and end at last in its proper key. Then the other parts are made from this, by taking to each note thereof its proper concord, according to the following rules. But sometimes a treble is first made, and a base put to it; and then you must reckon your concords downwards, as before you did upwards from the base. In composition, good air as well as good harmony ought to be maintained quite through the piece, to which all strict rules must give way. For in all cases the ear must be umpire of the sweetness of the music, which ought to be attended to before any servile fixt rules. The art of composing music in parts is called *descant*, but formerly *counterpoint*, whose rules here follow. In composing, strive to make one part imitate another in the notes. And in taking the concords, note must be set against note, according to the following rules; and the parts may be allowed to pass through one another. And in music of many parts, the upper parts ought to be joined together as close as possible.

Passages of Concords.

Rule 1. Two fifths or two eighths are not allowed to go together, in any two parts; for they are too luscious for the ear; and playing an eighth to the whole tune may as well be admitted. Yet they may be allowed when one part stands still, or rises or falls an eighth, or when they pass by contrary motion.

Rule 2. One may take as many concords (of one sort) as one will, the base standing still. Or may move to any concords.

Rule 3. One may change from any one, to any other concord; but more elegantly to some than
D d others,

Fig. others. Not so well from one perfect to another. Nor when the higher part skips to a perfect, and the base moves but a note. Nor when both skip to a perfect. Nor to rise with the base from a sixth to an eighth, and the contrary.

Rule 4. You may take as many thirds or sixths together as you will, especially if they are alternately greater and less. For the music is the best, where there is the most variety of concords.

Rule 5. Take such concords as are had with the least remove. Therefore the lesser sixth passes best into a fifth, and the greater sixth into an eighth.

Rule 6. Moving by leaps is best done into imperfect concords.

Rule 7. In a sharp key, a sixth is required to the second and third, both above and below the key, because the base commonly wants a third to its full compass.

Rule 8. When the parts move gradually, one may take two or three thirds and sixths, or a fifth and sixth by way of binding.

Rule 9. In composing several parts, set not the same cord twice over, or between any two different parts. For they must vary thus, 3d, 5th, 8th, or 4th, 6th, 8th.

Rule 10. When a passage is disputed, the ear must be judge. For by that *criterion* were all the rules found out.

Passages of Discords.

Rule 1. Discords are allowed in transition or breaking of notes; provided the leading note be a concord.

Rule 2. Discords are allowed in *binding notes* that is, when a note of one part breaks off in the middle

middle of a note of another part. These are called *Figured notes of syncopation, or driving notes*. Here a discord is allowed in half the binding note.

Rule 3. Discords must only be upon short notes, that soon pass the ear.

Rule 4. Discords are allowed when the base stands still, and the other parts move gradually; here they must needs fall upon discords.

Rule 5. Discords are allowed at a close, and are generally resolved into the nearest concords. Discords are prepared by concords, and resolved into concords. Thus a lesser 7th is resolved into a fifth. Although discords are disagreeable in themselves, yet when they pass into concords it makes the music very delightful; and is like a sweet cordial after a bitter potion.

Cadences.

Rule 1. The base intends a close, when it falls a fifth, after rising or falling one or two notes. And in falling a fifth (or rising a fourth), the former note requires a greater third. At a close the base is to fall a fifth.

Rule 2. The closes in any key, are the key, the 2d, and the 6th and 3d.

Rule 3. The closes in any key, fall naturally in the order set down in the last rule.

Rule 4. At a close, you may remove by leap to any concord.

Rule 5. At a close, discords are used both by leaps, and binding, promiscuously according to the former rules.

This mixture of concords and discords do very much delight the ear, and keep it in suspense, till perfect concord comes to compleat the harmony.

Though the closes naturally come in due course one after another, yet a judicious musician, by the

Fig. artful introducing flats and sharps, can skip over some and take others out of course; but then it is a little painful to the ear; and so, indeed, in some degree, is every shifting of the key.

There are several particular rules of the passages of concords and discords which I have not mentioned, but they all depend on what has been delivered.

In playing *figured* or *thorough base*, where nothing is marked, *common cords* must be played, which are the 3d, 5th, and 8th. The other cords are marked with their proper figures. A flat or sharp before among the figures, signifies a flat or sharp third to be played. To play a good thorough base, and manage discords rightly, one must know, 1. How to prepare them. 2. How to accompany them. 3. How to resolve them. Accompaniments are cords taken in, to fill up the harmony. But I have no design of meddling with the practical part of music; I have treated of it so far as mathematics have any concern in it, and I promised no more.

A R T. XVI.

A general Method of Philosophising; or, Rules to be observed in Philosophical Enquiries.

THE design of philosophy is to find out the nature and causes of things from their visible effects; and these effects are the phænomena which are made known to us by observation and experiments. In order to philosophize aright.

1. We must make a competent number of observations and experiments, whereby we may know that the effect we reason about is certainly true.

2. We must find out all the causes, as far as possibly we can, that can produce this effect; and the truth of them must be mathematically demonstrated, when necessary. This is the most difficult part of the business; for it is hard to know when we have got all these causes. And therefore a philosopher ought to have a very extensive view of things, and a thorough knowledge of the subject he enquires after. If we be not sure that we have enumerated all the causes, then the effect may proceed from some cause we have omitted; and so far the proposition will remain doubtful.

3. Sometimes it is necessary to shew and demonstrate what cause cannot produce the effect; especially when that effect has been vulgarly and erroneously attributed to such imaginary cause.

4. To proceed, we must compare all the possible causes of any two of these effects (or phænomena) and reserve such as are common to both. Again, compare these with all the possible causes of a third phænomenon, and reserve the causes that are common to both. In like manner compare these with

Fig. all the causes of a fourth phenomenon, &c. Proceed thus till there be but one cause at last common to both; then it is evident this is the cause sought after, for all the rest are exhausted.

5. After the true cause is found out, the same principle must be applied to the solving all phenomena of the same kind; and if the explications given thereby thoroughly agree, it will confirm the truth of the principle from whence these solutions are had. For if they be true, they must stand the rigour of mathematical calculation. But if they cannot do this, we must take it for granted that we have fallen upon some wrong principle; and therefore we must look out for some other. That this coincides with the Newtonian method of proceeding, I shall shew in the following Examples.

E X A M. I.

To find the cause of the revolution of the heavenly bodies, and of retaining them in their orbits.

Phænomena.

1. The satellites of Jupiter and Saturn, describe areas proportional to the times of description about their primary planets.
2. The primary planets describe areas proportional to the times of description, about the sun but not about the earth.
3. The periodic times of the satellites about the primary planets, or of the primary planets about the sun, are in the sesquiplicate ratio of the mean distances from their respective centers.
4. The moon describes areas round the earth proportional to the times of description nearly.

Propositions.

1. Every body that describes areas proportional to the times of description about a point

is urged by a centripetal force tending to that Fig. point.

2. If the periodic times of revolving bodies are in the sesquiplicate ratio of the radii; the centripetal forces will be reciprocally in the duplicate ratio of the radii.

3. If a body moving in a fluid revolve round a center, it will be retarded, and the area will not be as the time.

4. A spherical body revolving round its axis in an immense fluid, will cause the parts of the fluid to revolve, so as the periodic times will be as the squares of the distances, from the center of the sphere. And this will constitute a vortex; and bodies revolving in it must be of the same density, and observe the same laws of revolution.

5. If the moon is retained in its orbit by a force reciprocally as the squares of the distances. That very force at the earth's surface will make bodies descend 16.1 feet in a second.

Solution.

From phenomena 1st and 2d, the revolving body must be carried round, and continued in its orbit, either by some force perpetually acting towards the center, in free space. Or else it must be carried round in a vortex. For it cannot move in a resisting medium by Prop. III.

And by phæn. 3d, the revolving body is retained in its orbit by a force acting towards the center, which is reciprocally as the squares of the distances, by Prop. II. For it cannot be carried in a vortex by Prop. IV.

By phæn. 4th, and Prop. I. the moon is retained in its orbit by a force acting towards the earth, which is reciprocally as the square of its distance. And by Prop. V. this is the force by which bodies descend, called *gravity* or *weight*.

Fig. Now any other causes than these we have mentioned cannot be thought on. And therefore the true cause sought is discovered to be a gravitating force, with which all bodies in the universe are endued. And therefore the planets, comets, and all the satellites are retained in their orbits by this force, and all move in free space, or at least in a medium so extremely rare, as to make no sensible difference in a long space of time.

This cause or principle being found out, when duly applied, will explain all the phenomena relating to the moon's motions, the precession of the equinoxes, the motion of the sea, &c. as may be seen in the Principia, Lib. III.

EXAM. 2.

To investigate the cause of the refraction of light.

Phænomena.

1. If the sun's rays come through a hole in a window shut, and be refracted by a glass prism; an oblong image will be formed on a paper, or the opposite wall, consisting of several colours, whose length is perpendicular to the axis of the prism.

2. If another prism be placed behind the former, and with its axis perpendicular to it; it will refract the image sideways, of the same length as before, but of no greater breadth.

3. If a small part of the light of any one colour, be again refracted through one or more prisms, the image will always be round, and of the same colour as before.

Propositions.

1. If light consists of heterogeneous parts, which are differently refrangible; being refracted through a prism, it will paint an oblong image on the opposite wall.

2. If

2. If light consists of homogeneous parts, and Fig. passes through a prism, so that the angle of incidence and emergence may be equal in the two refracting planes; the image will be nearly circular.

3. If light be of such a nature, that its rays will split or dilate when refracted; then a prism will refract the light into such an image, that the breadth will be as great as the length; and this always, by all successive refractions; and the image will be enlarged by every refraction.

Solution.

From phæn. 1 and 2, it appears, that light is made up of heterogeneous parts, which are differently refrangible. For it cannot consist of homogeneous parts, by Prop. II. Nor does its rays split or dilate, by Prop. III. Moreover from phæn. 3, that small part of light, which continues of the same colour, must be homogeneous, by Prop. II. Hence light is discovered to consist of heterogeneous parts all mixt together, but capable of being separated from one another by refraction, into the compounding homogeneous parts. And the properties of these homogeneous parts are innate and essential to them, and cannot be altered by any reflections or refractions whatever. The nature of light thus found out, may be applied to explain other phænomena of the like nature.

EXAM. 3.

To find out the cause of the ascent of water in a syphon or pump.

Phænomena.

1. If the end of a syringe be put in water, or any fluid, and the piston drawn up, the fluid will follow and rise with the piston. Or if the mouth be applied to the top of a tube immersed in

Fig. in a fluid, sucking at the top will draw the fluid up.

2. Cement a brass cock in one end of a close tube, exhaust the air and shut the cock. Then put the end with the cock into the water, and open the cock, and the water will ascend into the tube.

3. Cement a syringe in the mouth of a glass bottle, quite full of water; and draw up the piston, but no water will follow.

Propositions.

1. If a vacuum be impossible in nature; then any fluid having free liberty, will move towards such a vacuous place to prevent it.

2. If there be a greater pressure upon one part of a fluid than on another; the fluid will move towards the part having the least pressure.

3. Whatever fluid is drawn or moved by the force of suction, will always be drawn or moved by that force when applied, if the way be clear.

Solution.

By phæn. 1 and 3, water ascends up a tube either by the pressure of an external fluid, or by nature's *fuga vacui*; for it cannot ascend by suction, by Prop. III.

By phæn. 2, the water ascends the tube by the pressure of an external fluid; for it is not by any *fuga vacui*, by Prop. I.

Hence water ascends into cyphons and pumps by the pressure of some external fluid; and we know of no such fluid but the air or atmosphere.

E X A M. 4.

To find whether the diurnal revolution belongs to the sun or to the earth.

Phænomena

Phænomenon.

Fig.

The body of the earth is never in the plane of the sun's diurnal orbit all the year; except at the equinoxes, when it passes through that plane.

Proposition.

Every body moving round another body as its center, to which it is continually drawn or impelled; must move in a plane passing through that central body.

Solution.

Hence the apparent diurnal revolution of the sun, proceeds from the real revolution of the earth round its axis. For if the sun did really move round the earth; then would the earth always be in the plane of the sun's orbit, by the Prop. But by the phæn. she is not so. Therefore the sun moves not round the earth.

The phænomenon is proved thus, the shadow of the end of a gnomon erected on any plane, describes always a curve line on that plane, except at the equinoxes.

EXAM. 5.

To find whether the annual revolution belongs to the earth or to the sun.

Phænomena.

1. The sun and planets don't describe areas round the earth proportional to the times.
2. The periodic times of the sun and planets round the earth, are not in the sesquiplicate ratio of their distances from the earth.
3. The earth and planets describe areas round the sun, which are as the times of description.
4. The periodic times of the earth, and other planets round the sun, are in the sesquiplicate ratio of their distances from the sun.

Propositions.

Propositions.

1. Bodies that describe areas porportional to the times of description, round another body; are urged by forces directed to that other body.
2. If several bodies revolve round another body, and their periodic times be in the sesquiplicate ratio of the distances, they are urged to that body by forces reciprocally proportional to the squares of the distances.

Solution.

By Prop. I. and phœn. 1, the planets don't revolve round the earth. But by phœn. 3, and Prop. I. the earth and planets revolve round the sun. Also by phœn. 2, 4, and Prop. II. the earth and other planets are urged by forces, reciprocally as the squares of the distances from the sun, but not from the earth. Therefore the annual motion belongs to the earth.

A R T.

A R T. XVII.

Optics, being a Translation of some Papers of Sir J. Newton's Optical Lectures, Part II. never Englished before, p. 239, &c.

BEFORE I proceed to another kind of experiments, it is necessary that we look more particularly into the form of the coloured image; which the light coming in at a strait round hole into a darkened room, and passing through a prism, paints there; and carefully take the dimensions of all the colours, and their distances from one another, and also the several degrees of refrangibility, corresponding to all kinds of rays.

It was shewn before, that when a prism (whose vertical angle is about 63 degrees) projects the image at the distance of 22 feet, its length was $13\frac{1}{2}$ inches, and breadth $2\frac{1}{2}$. And therefore the centers of the outmost circles, of which disposed in length, that image consists, were distant $10\frac{1}{2}$ inches. Now to this distance or contracted longitude of the image it is proper to refer the other dimensions thereof, because they have no certain relation to its absolute length (which depends on the magnitude of the compounding circles). And by this I could investigate with more exactness the places where the colours were the most perfect in their kind; and their confines, falling upon a cross paper, which I noted with a pen, and by many repeated observations of this sort compared together; I at last gained these particular conclusions.

1. The blue and violet on one hand, and the green and red on the other; take up on each hand half the image; so that the confine of the green
and

Fig. and blue (which I may call a sea-green) lies in the middle.

2. The place where the leeky or brightest green appeared, divided the contracted length of the image in the ratio of 3 to 5; so that the length being divided into 8 parts, that green was 3 parts distant from the extremity of the red, and 5 parts from that of the purple.

3. The space taken up by all the green, as far as the confines of the blue and yellow, was about the sixth part of the whole contracted image.

4. The confine of the blue and purple, or the most perfect indigo, was distant from the confine of red and yellow, or the most perfect orange, about $\frac{7}{13}$ of the whole contracted image.

5. Lastly, this distance of the indigo and orange, was divided by the confine of green and blue in the ratio of 2 to 3; so that this confine, or the middle of the image, was distant from the indigo $\frac{1}{6}$ parts of the whole contracted length, and $\frac{2}{3}$ parts from the orange.

When I had observed these things as carefully as I could, not altogether trusting to my own senses, but (on account of the great difficulty of precisely distinguishing the confines of the colours, and the places of greatest perfection) taking the judgment of other people; I drew out the dimensions of the image according as I had found them; as may be
88. seen in the figure. That is, with the centers X and Y $10\frac{5}{8}$ inches distant, and with the semi-diameters $1\frac{5}{8}$ inches, I described two semi-circles APC and BTD, and drew the tangents AB, CD.

Then I divided the line XY (which I before called the contracted length of the image) into 60 equal parts, and made LY = 9, IY = 20, HY = 30, and FY = 44 parts thereof. And erecting perpendiculars at these points, I distinguished the image into five parts, corresponding to the five
more

more eminent colours; the part PF shewing the Fig. space of the violet, and FH the space of the blue, 88. etc. which being done, I projected the coloured light upon this figure, that it might again appear, whether every colour continued still within the assigned limits; and when the whole image filled the whole figure, the several colours with their several parts very well agreed; and in the mean while, I observed the places in these spaces (which were marked out by points in this scheme) where the several colours in their kinds appeared full and brightest.

Now it is evident there can be no other intervals of these places and limits, terminating the colours; although, by the methods often mentioned before, you never so much diminish the circles, which compose the image, keeping still the same centers. And for this reason, that the heterogeneous rays will be more separated, and the colours come out more simple. For since, in the very terminating right lines AB and CD, the colours are quite simple; and the colours in the middle of the image near the line XY, appear to be of the same kind with those lying between that and the margin; reason tells us, that a mixture of heterogeneous rays does not sensibly change the place of any colour, since their coming from this side and that, they temper one another. Thus the green and purple rays are equivalent to the blue, and therefore when mixt with it, don't alter that colour; but it appears the same as if no purple rays had been mixt therewith. But here we must except the terminating circles AC, BD, where the temperament on the outside gradually decays, and becomes of a full red, on one side, where it runs into the terminating circle. But if any one should 89. pick here, he may try new experiments, and contract the breadth of the image, that the circles may be

Fig. be less, the rest remaining; and I doubt not but 89. every thing will answer.

But although the confines of the colours, fell the lines passing through the points F, H, I, L yet the places where they appear the most full and intense, do not always fall in the middle of the included space; for the blue, which in its kind was the brightest, and no way tinged with purple, fell nearer F than H, and the intensest yellow was a little nearer L than I. And thus the red and purple appeared intense a little nearer the centers X and Y, than to the other limits; only the green was brightest in the middle between the limits F and H. Whence appears the reason, that although the yellow and blue by their mixture compose green, yet the red and green, by reason of the too great distance, don't well compose a yellow nor the green and purple a blue. Therefore seeing the colours are more close about the middle, so as between the yellow and red, as likewise between the blue and purple, there is about a third part more interval than between the green and yellow or blue, terminated on both sides; by this the image is elegantly distinguished into parts, proportional among themselves, containing five principal colours, and two more which we shall take in, and these are, orange between the red and yellow, and indigo between the blue and violet; and that because next to these five principal colours, these two eminently appear, and the spaces assigned them are large enough for the degree of perfection of each and thus the too great expansion of the outer adjoining colours is shortened, and they are all made in a neater proportion to the quantity of the green.

These colours being inserted, I began again to make observations, and (in a word) all things appeared, as if the parts of the image, which contain the colours, were proportional to a string of

vide

vided, so as to give the sound of the several degrees or notes in an octave. When I found this, 89.

I divided the figure of the image into like parts, as may be seen in the scheme. And then I tried how well the colours agreed with these parts. Thus, produce the contracted length of the image XY to Z, so that $YZ = XY$, and suppose the string to be XZ, and that XY be so divided, that all the segments extended to Z, may sound the several notes of an octave (*sol, la, fa, ut, re, mi, fa, sol*). And that is done by bisecting XY in H, and trisecting it in G and I, and again trisecting XI in F, and taking KY a fifth part, and MY an eighth part of XY. And the semi-tones FG and KM will represent the indigo and orange; and the other five tones XF, GH, HI, IK, MY, the five eminent colours; all of which in particular, and the whole train of colours, falls exactly into the whole figure, and were contained respectively in the several parts. And about the middle of these parts, each colour in its kind appeared the brightest and most intense; even the purple and red, beyond P and T, abounded with light, gradually vanishing.

But I could not observe these things so precisely, but I must confess they may be contrived a little different. To which purpose, if there be taken eleven mean proportionals, between XZ and YZ, of which FZ is the second, GZ the third, HZ the fifth, IZ the seventh, and KZ the ninth; then this division of the image seems to agree very well with the expansions of the colours. For such minute differences as arise between this and the former division, can hardly be accounted errors by the judgment of the most acute sense.

How these divisions differ will appear by the adjoining numbers, where the former express the parts of the cord, divided in the musical ratio; and

Fig. the lower, the like parts, divided in a geometrical
89. ratio. The card being divided into 360 parts.

360, 320, 300, 270, 240, 216, $202\frac{1}{2}$, 180, (by
musical division) and

360, 321, 303, 270, 240, 214, 202, 180, (by
geometrical division).

And I chused to lay down the former division,
not only because it agrees very well with the phœ-
nomena, but because it may perhaps contain some-
thing about the harmony of colours, analogous to
the concords of sounds, which the painters are not
quite ignorant of, but I am unacquainted with.
Wherefore it seems likely to a considering person,
that the affinity which is between the outside pur-
ple and the red, the extremities of the colours, is
found likewise between the terms of an octave,
which in a manner give the same sound.

Hence at last the proportions of the fines of re-
fraction, belonging to all kinds of rays, are (me-
chanically) determined. Since in glass contiguous
to air, the fines of the extreme rays are as 68 and
69; divide the intermediate 1, in the ratio of the
parts of this image, and there will arise 68, $68\frac{1}{2}$,
 $68\frac{1}{3}$, $68\frac{1}{4}$, $68\frac{1}{5}$, $68\frac{1}{6}$, $68\frac{1}{7}$, 69, for the fines at the
confines and limits of the seven colours, in respect
of the common fine of incidence $44\frac{1}{2}$, when the
refraction is made out of glass. But when it is
made into glass, for these fines take the numbers,
68, $68\frac{2}{3}$, $68\frac{1}{3}$, $68\frac{1}{2}$, $68\frac{2}{5}$, $68\frac{3}{5}$, $68\frac{4}{5}$, 69; the
common fine of incidence being 106. And for
the fines for those rays, where the colours are the
most perfect in their kind, intermediate numbers
may be found between these.

Thus in water contiguous to air, where the ex-
treme fines of refraction are as 90 to 91, one may
find the intermediate fines by a like dissection of
the intermediate unity (by making them as 90, $90\frac{1}{2}$,
 $90\frac{1}{3}$, &c. or 90, $90\frac{2}{3}$, $90\frac{1}{3}$, &c.) But note, these

divisions

divisions are not precisely geometrical, but yet as Fig. nearly accurate as any practice requires; and to toil 89. any further, besides the tediousness of computation, would prove nothing but an affected and vain curiosity.

There are other circumstances concerning these colours, which I could now determine; such as the various forms and expansions thereof, for the various positions of the prism revolving about its axis; or for the various refractive matter, of which the prism is made, and with which it is encompassed; and also for the various magnitude of the vertical angle. But all these things will appear plain enough, from what is shewn in the first part, comparing them with the foregoing explications; so that all effects, as far as I know, may be found out, either for one single refraction, or for more, and for any termination of light.

I have corrected a great many errors in this paper; how they have happened I do not know.

A translation of part of a paper in the optical lectures, intituled thus. Of the phenomena of light passing through a refracting medium, terminated by parallel planes. Part II. Sect. 2. p. 261, &c.

Having gone through the phenomena of triangular prisms, those made by quadrangular ones, which are composed of parallel plains, come now properly to be spoken of. And I the more willingly engage in it, as the philosophers have hitherto believed, that no colours can be generated after this manner, supposing that the latter surface destroys all the effects of the rays by a contrary refraction, to what the former produced; and they take it for a matter of experience, because in glass windows, and such like, they see none produced. But in this they are deceived, because the quantity

Fig. and perfection of the colours, depends upon the distance of the parallel surfaces. For in glass plates, by reason of the small distance of the surfaces, the colours are very thin and weak, and contained in such a narrow space, as to escape the senses; but in glass of a greater thickness, or rather in glass vessels or parallelopipedons full of clear water, the colours are plainly seen to be generated.

90. For let ABCD be a glass, or water parallelopipedon, inclosed with air, whose parallel surfaces are denoted by the two lines AC and BD. And let the sun shine through it, passing obliquely through the small hole F. Then the rays, by the former surface at H, ought to be so unequally refracted, that they will diverge from one another, till they fall upon the latter surface at PT, and there paint all kinds of colours, as has been abundantly explained before. Now since by the parallelism of the refracting surfaces, the rays are only bent at the latter surface, just as much as they were at the former. It is necessary that they emerge in the air parallel to one another, and to the incident rays SH; and therefore the distances and positions they have acquired, they will preserve ad infinitum, and the colours produced will always remain the same without any variation. Wherefore if HP be the purple ray, and HT the red ray, refracted at H, the rays Pp, Tt, being again refracted, will emerge parallel to the incident rays SH, and to one another; and therefore they transfer the purple and red, which they give at P and T, to any distance unchanged, and preserve them every where without any variation; the purple being translated from P to p, and the red from T to t; and the rest of the rays into their correspondent intermediate places.

And this must exactly happen when the rays fall upon this prism in lines parallel to the same line

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SF, and they emerge parallel; but when they are Fig. inclined to one another, as happens when they pro- 90. ceed from different parts of the solar disk; then indeed they emerge inclined to one another, and in their last translations they undergo some change. Because the circles affected with all kinds of rays, by which (being disposed in length) the coloured image of the sun is made, by falling on the surface BD; by reason of the diverging of the rays, crossing one another in the hole at F, come out the more dilated, as the rays are further from the point of emergence; whilst their centers, which are illuminated by rays from the sun's center, moving in any direction before refraction, after refraction preserve the same distances and positions for ever. And hence it is, that the space *pri* illuminated by the solar light immitted into a dark room, is the more dilated by that, and drawn into an orbicular form, as its termination is further from the prism; and the greenness in the middle R, if there is any, is gradually changed into whiteness; but if there is none but by the narrowness of the prism, or wideness of the hole letting in the light, the whiteness possesses the middle of the colours; the same whiteness is sensibly dilated. But yet the colours are not thereby dilated, nor drawn into a lesser space, however less luminous they appear by the dilatation of the image.

If we view any visible objects through this parallelopipedon, they are no otherwise tinged with colours, than as a triangular prism, especially if the parallelopipedon be oblique enough to the transient rays, that it may refract much, and that the objects be sufficiently near. For if the objects are far off, whether that distance lies between the parallelopipedon and objects, or the parallelopipedon and the eye; how great soever the refraction is, by the obliquity of the parallelopipedon, yet

Fig. no colours are produced. If X be a lucid point, 91. sending rays to the eye at S, through the refracting planes AC and BD of the parallelopipedon; it is plain, that drawing SNX the path of the red ray, and $SpMX$ that of the purple ray, these rays are equally refracted at both surfaces; and therefore the triangles pSt , MXN are similar; the purple ray, by reason of the greater refrangibility on both sides at p and M , deviating more from the direct path than the red; whence they must needs cut one another within the prism, as appears at O , making again the triangles pOt , MON , similar, or the trapezium $SpOt$ similar to the trapezium $XMON$, and therefore they come to the eye, as if they had at first proceeded from the same point O , and had only been refracted at one of the surfaces. And hence it follows, not only that the colours will be generated, but also that the angle pSt , or the apparent breadth of the colours, and other circumstances, may be determined for any position of the eye. And therefore it will be plainer, by making experiments, that objects deeply immersed in water, will appear a little tinged with colours, by the refraction of the upper surface, to an eye beholding them obliquely. For AC may represent the surface of the stagnant water, and O any object immersed in it, which the spectator at S beholds. And O is easily found by drawing the right line SX , cutting the refracting surfaces in K and L ; and dividing it in O , so that SK be to LX as SO to OX , or as KO to OL .

A translation of part of the last paper in the optical lectures, concerning the rainbow, p. 285, &c.

There still remains that wonderful spectacle of the celestial arch, to the explication of which *Cartes* has paved the way; for to him it is owing, that we know that it is formed in the drops of falling rain; which

which is plain from this, that it is never seen but Fig. in rainy weather, where the sun illuminates the falling rain; and appears sometimes in places very near, as if it was not placed in the air, being fixt upon the walls of the opposite houses, or rather in the intermediate space; that water by some sort of artifice being thrown out in drops shows the rainbow, and the grass in a dewy morning, full of small drops, gives the colours of the rainbow. To him is also owing the most ingenious invention of the refractions of a drop, and the limits thereof, but the physical cause he did not so happily find out. That you may understand this, suppose a ray AN to fall upon the globe NFG at N, and be thence refracted towards F, where again it is either refracted towards V, or perhaps reflected to G, 92. And if it afterwards emerge, it is refracted again to R, or reflected to H, and so on; so that some of the rays entering the globe, as NFV presently go out, suffering no reflection; others as FGR, after one reflection; and others as GHS after two; others after three, or even more. Now since the drops of rain, in regard to the distance from the eye of the spectator, are exceeding small; they may be taken for physical points; and therefore there is no occasion for considering their magnitudes, but only the angles which the incident rays make with the emergent ones; for where these angles are the greatest or least, the emergent rays are each of them denser; and because for divers sorts of rays, there are also divers of these greatest or least angles; each of them thus dense tending to different points, will prevail and exhibit their proper colours. Therefore we must determine these greatest or least angles, which the emerging rays of all sorts, contain with the incident rays, that we may rightly perceive the reason of these phenomena.

Fig. It is shewn in Cor. 1. and 2. Prop. 35. (Part. 1.)
 92. that the emergent ray GR is the least inclined to the incident ray AN, when $3RR : II - RR : CN^2 : ND^2$. And $I : 2R :: ND : NE$, putting I to R as the sine of incidence to the sine of refraction; and ND and NE being thus found, GR will be given in position.

For example, in the most refrangible rays, the sine of incidence is to the sine of refraction, or I to R, as 185 to 138, as is found the nearest in rain water; and it will be, $57132 : 15181 :: (3RR : II - RR :) CN^2 : ND^2$. Therefore $DN = \sqrt{\frac{15181}{57132}} \times CN = \frac{5155}{10000} CN$. Whence by the table of sines, there is given the arch NL 62 deg. 4 min. Further, since $I : 2R :: ND : NE :: 185 : 276 :: \frac{5155}{10000} CN : NE$, NE will be $= \frac{7691}{10000} CN$.

And then by the table of sines there is given the arch NF 100 deg. 32 min. Then subtract twice the arch NF from the sum of the arches NL, and 180 deg. or a semi-circle, and there will remain 41 deg. 0 min. for the inclination of the rays RG and AN, or for the angle AXR; that is, producing AN and NG to meet in X, and this is the angle under which the inner or blue border of this bow ought to appear, or its least semi-diameter.

After the same manner for the least refrangible rays, putting the sine of incidence to the sine of refraction, as 182 to 138, as I measured them, there will be found $ND = \frac{5028}{10000} CN$, and NE

$= \frac{7533}{10000} CN$; and thence by the table of sines,

NL will be 60 deg. 22 min. And the arch NF 98 deg. 38 min. And therefore the angle AXR 43 deg. 6 minutes; under which the external, or

red edge of the bow, will appear. Therefore its Fig. greatest semi-diameter is 43 deg. 6 min. From 92. which take the least semi-diameter 41 deg. 0 min. there remains the breadth of the bow about 2 deg. 6 min. Or rather, by adding the sun's diameter (which is 31 min.) 2 deg. 37 min. But as the colours at both edges of the bow, are so weak as not to be seen for the splendor of the adjoining clouds, its breadth as to sense, scarce exceeds 2 degrees.

The dimensions of the exterior bow are determined in a manner no way different. For it is shewn in Cor. 1. and 2. Prop. 36. (Part. I.) that the inclination of the emergent and incident rays HS and AN, is the greatest, when it is $8RR : II - RR : : NC^2 : ND^2$. And $I : 3R : : ND : NE$. Wherefore putting the numbers 185 and 138 for the sines of the most refrangible rays I and R, as before; we shall get $ND = \frac{3157}{10000} CN$, and NE

$= \frac{7064}{10000} CN$, whence, by the table of sines, the arch $NL = 36$ deg. 48 min. and the arch $NF = 89$ deg. 53 min. And therefore the angle $AYS = 52$ deg. 51 min. which will be the greatest semi-diameter of this bow. And likewise for the sines of the least refrangible rays, I and R, putting the numbers 183 and 138, there comes out $ND = \frac{3079}{10000} CN$, and $NE = \frac{6965}{10000} CN$. Whence by

the table of sines, we shall get the arch $NL = 35$ deg. 52 min. And the arch $NF = 88$ deg. 18 min. And therefore the angle AYS will be 49 deg. 2 min. the least semi-diameter of the bow. Therefore if from the greatest semi-diameter 52 deg. 51 min. there be subtracted the least 49 deg. 2 min. and to the remainder be added the sun's diameter, 31 min. The breadth of this bow will come

Fig. come out 4 deg. 20 min. But by reason of the
92. greater obscurity of this than of the interior bow,

I imagine the colours will scarce appear further than the breadth of three, or three and a half degrees.

That I may distinctly represent to view the reason of these bows, let E, F, G, be the drops any way thrown into the air. SE, SF, SG, the parallel rays of the sun falling on these drops; EM, EN, EO, the different refrangible rays, coming out of the drop E, after one reflection; and FN, FO, FP, and also GO, GP, GQ, like rays coming out of the drops F and G; that is, EO, FP, GQ, the most refrangible rays, and EM, FN, GO, the least refrangible, &c. Now if the eye of the spectator be placed at O, it is plain from the hypothesis, that among the rays which the drop E emits after one reflection, only the most refrangible or blue, as EO fall upon the eye; the rest as EN and EM, passing by from the lesser refraction; and therefore the blue colour is seen at E. But among the rays which G emits after one reflection, the most refrangible, as GQ, pass by the eye, because they are parallel to the ray EO; and the rays of another sort, suppose the least refrangible, or red rays as GO, fall upon it; whence the red appears at G; and by the like argument the drop F, in the middle between E and G, sends the mean refrangible rays, as FO to the eye; the rest as FN, FP, passing by on each side; and therefore the green is seen at F. The reason is the same for all drops placed at the same apparent distances from the axis OR, which passes through the sun and the eye; and therefore the colours will every where appear at these distances. That is, this coloured bow, whose inner side is blue, and outer side red, and the middle parts of the middle colours, will then appear, the angle OGQ or GOE remaining; and the breadth of this bow will be about two degrees,

rees, as I have shewn before; and the like rea-
soning holds good concerning the outer bow; ex-
cept that the order of the colours is contrary, by
reason of the contrary inflection of the rays. But
the drops which are situated out of these bows on
any side, can throw no rays at all upon the eye,
after one or two reflections, and two refractions;
or on the other side, they are all mixt together,
and become insensible; and therefore can exhibit
no phænomena of that kind; but the sky in these
places will appear as usual.

Besides the phænomena of colours of which we
have been treating, there are still many other things
especially about the colours of very thin transpa-
rent plates, such as bubbles of water, and air com-
pressed between two planes, and the very thin skins
of several things), the cause and measure of which
can hardly be accurately determined without ma-
thematical reasoning; but in these things I seem to
have enlarged too much, and therefore I now be-
take myself to the more abstracted parts of the
mathematics. The end.

Fig.

A R T. XVIII.

A Collection of curious, useful, and important Problems.

P R O B. I.

Given an equation of x and y , as $x^a y^b + x^c y^d + x^e y^f + x^g y^h = 0$; to find how many ways it may be transformed; that there may always be one term without x , and another without y .

1. **D**IVIDE all the terms by x^a , then the equation is $y^b + x^{c-a} y^d + x^{e-a} y^f + x^{g-a} y^h = 0$, which is one way where one term is without x . In like manner, divide all by x^c , and the second term will be without x ; which is a second way. Again divide all by x^e , and the third term will be without x . Also divide all by x^g , and the fourth term will be without x . Hence the equation admits of as many different forms as there are terms, wherein there are always one term without x .

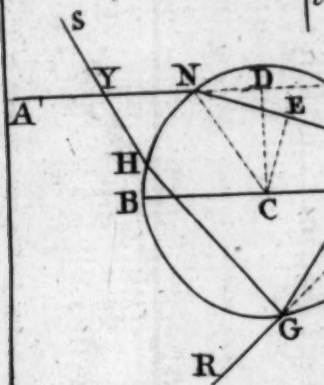
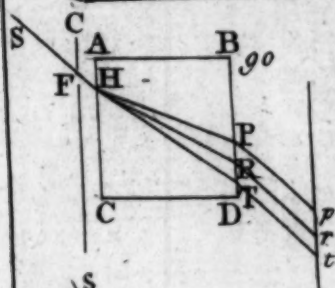
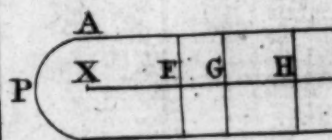
2. Again, in the equation $y^b + x^{c-a} y^d + x^{e-a} y^f + x^{g-a} y^h$, setting aside the first term y^b , the equation $x^{c-a} y^d + x^{e-a} y^f + x^{g-a} y^h$ will admit of as many forms, where one term wants y , as there are terms left, as appears by proceeding as in Art. I. Therefore if t be the number of terms in the given equation; the whole number of forms, where one term is without x , and another without y , is $= t \times t - 1$.

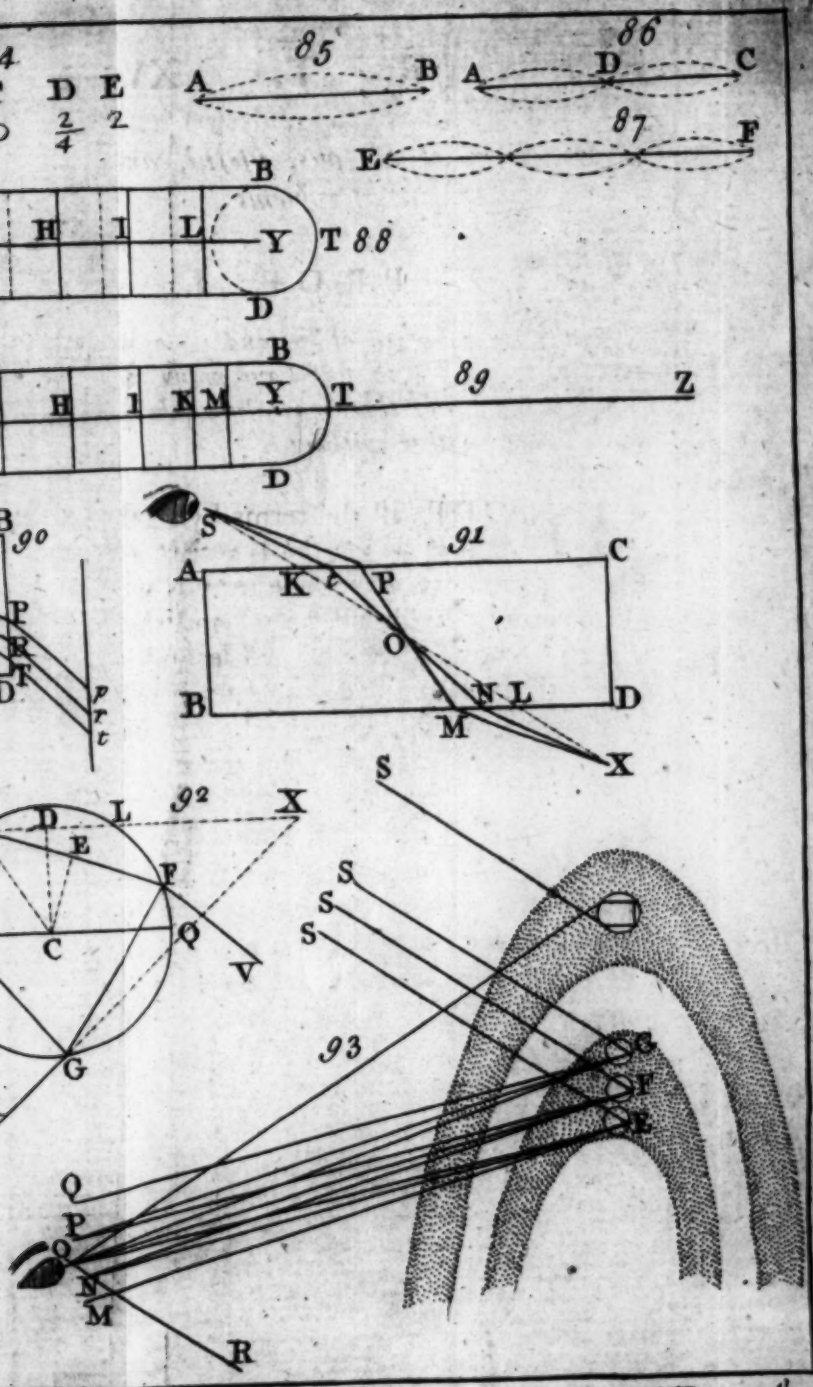
S C H O L.

This supposes that all the indices are different from one another; but if several indices be the same,

Fig 84

A	B	C	D
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$





same, the number of transformations will be less. Fig. Therefore in such cases, the number will never amount to $t - 1$.

PROB. II.

A and B have each a number of shillings; and A's number is to B's, in the ratio of $17.501501501 \&c.$ ad infinitum, to $31.917171717 \&c.$ ad infinitum; and they are the least in that ratio: what are the numbers?

By Prop. LVI. Chap. III. B. II. *Arithmetic*, the circulating part $501 \&c. =$ vulgar fraction $\frac{501}{999}$.

Also the circulating part $17 \&c. = \frac{17}{99}$. There-

fore $17.501501 \&c. = 17 \frac{501}{999}$; and $31.917 \&c.$

$= 31.9 \frac{17}{99}$. Therefore A's number is to B's, as

$17 \frac{501}{999}$ to $31.9 \frac{17}{99}$, or as $153 + \frac{501}{111}$ to $287.1 +$

$\frac{17}{11}$ or as $\frac{17484}{111}$ to $\frac{3175.1}{11}$; that is, as 1923240 to

3507378 , which in the least numbers will be as 320540 to 584563 .

PROB. III.

A usurer lends out $2000l.$ which the borrower is to clear off by quarterly payments, viz. $1l.$ at the end of the first quarter, $2l.$ at the end of the second, $3l.$ at the end of the third, and so on. In what time will the debt be cleared, reckoning 5 per cent. comp. interest for all.

Let $p = 2000$ the principal, $r = 1l.$ and its interest at a quarter's end, $e =$ interest alone, $x =$ time

Fig. time required; then the money due at the end of the several quarters will be as follows.

quarters expired.	money then due.	by substitution.	by restitution.
0	p		
1	$rA-1$	$= A$	$= rp - 1$
2	$rB-2$	$= B$	$= rrp - r - 2$
3	$rB-3$	$= C$	$= r^2p - r^2 - 2r - 3$
4	$rC-4$	$= D$	$= r^3p - r^3 - 2r^2 - 3r - 4$
5	$rD-5$	$= E$	$= r^4p - r^4 - 2r^3 - 3r^2 - 4r - 5$
&c.	&c.	&c.	&c.
x	$rG-x$	$= H$	$= r^xp - r^{x-1} - 2r^{x-2} - 3r^{x-3} - 4r^{x-4} \dots x$

Then *per* quest. H or $r^xp - r^{x-1} - 2r^{x-2} - 3r^{x-3} - 4r^{x-4} \dots x$
&c. = 0,

that is, $r^x \times : p - \frac{1}{r} - \frac{2}{rr} - \frac{3}{r^3} - \frac{4}{r^4} \dots$

to x terms = 0.

But by Prop. 30. Increments (Schol.) the sum of x terms of the series $\frac{1}{r} + \frac{2}{rr} + \frac{3}{r^3} \dots$ =

$$\frac{r}{r-1} - \frac{1}{r^x} \times \frac{x}{r-1} + \frac{r}{r-1} = \frac{r}{ee} - \frac{ex+r}{eer^x}. \text{ Therefore } pr^x - r^x \times \frac{1}{r} + \frac{2}{rr} + \frac{3}{r^3} \dots$$

$$\&c. = pr^x - \frac{r^{x+1}}{ee} + \frac{ex+r}{ee} = 0, \text{ and } peer^x -$$

$$rr^x + ex + r = 0, \text{ reduced } r^x = \frac{r+ex}{r-pee}; \text{ and } x =$$

$$\log. r = \log. \frac{r+ex}{r-pee}, \text{ and } x = \frac{\log. \frac{r+ex}{r-pee}}{\log. r} =$$

$$\log. \frac{1.47119 + .017233x}{.005395}; \text{ which is easily found by}$$

by trial and error. Here $r = 1.012273$, and $e = .01227$, and x will come out 85.3 ; so that after $85\frac{3}{4}$ quarters the debt will be paid, that is in $21\frac{3}{4}$ years.

SCHOL.

If the question had been to find when the debt is the greatest possible; then the two succeeding quarters H and K are equal; that is, $H = K$

$$rH - x + 1, \text{ whence } \frac{x+1}{e} = H = r^x \times p - \frac{1}{r}$$

$$- \frac{2}{rr} - \frac{3}{r^2} \text{ \&c. to } x \text{ terms} = pr^x - \frac{rr^x}{ee} +$$

$$\frac{x+1}{ee}. \text{ And } ex + e = peer^x - rr^x + ex + r, \text{ and}$$

$$r - pee \times r^x = r - e, \text{ whence } r^x = \frac{r-e}{r-pee} =$$

$$\frac{1}{r-pee}; \text{ and } x = \frac{-\log. r-pee}{\log. r} = 27.8; \text{ there-}$$

fore in near 28 months, or 7 years, the debt is the greatest.

PROB. IV.

A lends B 2000*l.* at 5 per cent. simple interest, and is to pay it back, thus, 1*l.* at 1 year's end, 2*l.* at 2 years end, 3*l.* at 3 years end, &c. To find when the debt will be greatest, when it will be 85*l.* 15*s.* and when all paid. (Lad. Diary, 1769.)

Put x = time, r = interest of 1*l.* = .05; then at the end of x years, he would owe $2000 + 2000 \times rx$, if he paid nothing. But since he pays 1, 2, 3, &c. pounds in x years, that with its interest must be deducted; which is $1 + r \times x - 1 + 2 + \frac{2r \times x}{x-2}$

Fig. $x-2 + 3 + 3r \times x-3$ &c. to x terms. But by arith. progression,

$$1 + 2 + 3 \dots x = \frac{x+1}{2} x. \text{ And}$$

$$1 + 2 + 3 \dots \text{to } x-1 : \times rx = \frac{x \times x-1}{2} rx, \text{ and}$$

$$1 + 4 + 9 \dots x-1^2 : \times r = x-1 \times \frac{x}{2} \times$$

$$\frac{2x-1}{3} = \frac{2xx-3x+1}{6} rx, \text{ by Exam. 7. Prop}$$

10. diff. method.

Therefore at the end of x years he owes

$$2000 + 2000 rx - \frac{x+1}{2} x - \frac{xx-x}{2} rx +$$

$$\frac{2xx-3x+1}{6} rx; \text{ that is, putting } s = 2000, \text{ and}$$

$$\text{reducing } s + srx - \frac{rx^3 + 3xx + 3 - r.x}{6} = \text{the}$$

debt at x years end. Therefore *per* quest.

$$1. s + srx - \frac{rx^3 + 3xx + 3 - r.x}{6} = \text{max.}$$

Fluxions,

$$sr\dot{x} - \frac{rx^2\dot{x}}{2} - x\dot{x} + \frac{3-r}{6}\dot{x} = 0, \text{ whence } r =$$

46 very near.

$$2. s + srx - \frac{rx^3 + 3xx + 3 - r.x}{6} = \pm 85/.$$

And the roots extracted $x =$ something less than 94, in the first case; and something more than 94 years, in the second.

$$3. s + srx - \frac{rx^3 + 3xx + 3 - r.x}{6} = 0, \text{ and}$$

extracting the root, $x = 94$ years very near, when all the debt is paid.

PRO

PROB. V.

To find the value of $\sqrt{a + \sqrt{b + \sqrt{a + \sqrt{a}}}}$
&c. ad infinitum.

Let x = that value, then $xx = a +$
 $\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a}}}}$ &c. and $xx - a =$
 $\sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a}}}}$ &c. = x the first va-
 lue, because it reverts to the same quantity; then
 $xx - x = a$ and $x^2 - x + \frac{1}{x} = a + \frac{1}{x}$, and $x - \frac{1}{x}$
 $= \sqrt{a + \frac{1}{x}}$, and $x = \frac{1}{2} + \sqrt{a + \frac{1}{x}}$.

PROB. VI.

To find the value of $\sqrt{a \times \sqrt{b \times \sqrt{a \times \sqrt{b}}}}$ &c.
ad infinitum.

Let x = its value, then $xx = a \times$
 $\sqrt{b \times \sqrt{a \times \sqrt{b}}}$ &c. and $\frac{xx}{a} = \sqrt{b \times \sqrt{a \times \sqrt{b}}}$,

and $\frac{x^4}{aab} = \sqrt{a \times \sqrt{b \times \sqrt{a \times \sqrt{b}}}}$ &c. = x ,

whence $\frac{x^4}{aab} = x$, and $x^3 = aab$, and $x = \sqrt[3]{aab}$.

SCHOL.

These two Problems and their solutions are *ŷ. Bernoulli's*; but the solutions are certainly wrong, for both the quantities are infinite.

PROB. VII.

To find the specific gravity of a fluid:

Take two bodies A, B, of unequal specific gra-
 vities,

F f

Fig. vities, but both greater than that of the fluid; make them of equal weights in vacuo. Suspend them by threads to a pair of scales as usual, and immerse them both in the fluid, and get the difference of the weights. Then this difference of weight = weight of a quantity of the fluid, equal to the difference of the magnitudes of the bodies A, B.

For let A be the bulk of that specifically heavier, B that of the lighter; x = what A loses in the fluid, y = what B loses. And when they are immersed, (put w = weight of A or B, d = diff. weights in the fluid.)

$$w - x = \text{A's weight in the fluid.}$$

$$w - y = \text{B's weight in the fluid.}$$

Then since B is bigger, it loses more than A, and A is heavier in the fluid; whence $w - x = w - y$, or $y - x$, or d = difference of the weights of the bulks of water A and B; that is, d = a bulk of water = $B - A$.

Or thus.

If two such bodies be made in equilibrio in the fluid; and weighing them in vacuo; the difference of the weights will be equal to the weight of a quantity of the fluid, equal to the difference of the bulks of the two bodies.

Note, the more unequal the specific gravities of A and B are, the more exactly that of the fluid will be found.

P R O B. VIII.

Two known metals being mixt together, as gold and copper, to find the quantity of each.

Let magnitude of the mixture = A, sp. grav. = a
 of the copper = B, sp. grav. = b
 of the gold = $A - B$, sp. grav. = g
 Then

Then the weights of

$$\begin{aligned} \text{the gold} &= gA - gB \\ \text{copper or allay} &= bB \\ \text{mixture} &= aA \end{aligned}$$

Then $gA - gB + bB = aA$, reduced $B = \frac{g-a}{g-b} A$.

Here the magnitude is known, by the weight of as much water, which is = absolute weight — weight in water.

Cor. *Weight* $A : \text{weight } B :: (Aa : Bb ::) a \times \frac{g-b}{g-b} : b \times \frac{g-a}{g-b}$.*Or thus.*

Let x = weight of gold,
 y = weight of copper,
 a = weight of the compound,
 b = weight of the compound in water,
 c = specific gravity of gold,
 d = specific gravity of copper,

then $c : 1 :: x : \frac{x}{c}$ = weight of water of the bulk of x . $d : 1 :: y : \frac{y}{d}$ = weight of water of the bulk of y .then $x - \frac{x}{c}$ = weight in water of the gold, $y - \frac{y}{d}$ = weight in water of the copper.whence $x - \frac{x}{c} + y - \frac{y}{d} = b$, or $\frac{c-1}{c}x + \frac{d-1}{d}y = b$.

$$\frac{d-1}{d}y = b.$$

therefore $\frac{c-1}{c}x + \frac{d-1}{d}y = b$, but $y = a - x$.
F f 2 therefore

Fig. therefore $\overline{c-d} \cdot dx + \overline{d-c} \cdot c \times \overline{a-x} = bcd,$

$$\text{reduced } x = \frac{bcd - acd + ac}{c-d} = \frac{a-d \times \overline{a-b}}{c-d} c.$$

$$\text{and } y = \frac{c \cdot \overline{a-b} - a \cdot d}{c-d}.$$

Cor. Let $g = \text{specific gravity of the mixture, then}$

$$g = \frac{a}{a-b}, \text{ and } a-b = \frac{a}{g}, \text{ therefore } x = \frac{g-d}{c-d}$$

$$\times \frac{ac}{g}, \text{ and } y = \frac{c-g}{c-d} \times \frac{ad}{g}. \text{ Whence } \overline{c-d} \cdot g :$$

$$a :: \begin{cases} \overline{g-d} \cdot c : x. \\ \overline{c-g} \cdot d : y. \end{cases}$$

PROB. IX.

If a piece of ground is to be planted with trees, so that they may be at a certain distance from one another. To find the order they must be planted in, to have the most trees possible.

Since there are only 3 regular figures which can fill a space, viz. equilateral triangles, squares, and hexagons; therefore if the trees be planted in any regular order, it must be in the centers of some of these. Now let $r = \text{radius of a circle wherein a tree stands, and spreads itself to that distance every way.}$ Now whatever figure a tree stands in, the inscribed circle must be the same in all of them, whether a triangle, square or hexagon. Therefore we must enquire what proportion each of these figures have to the inscribed circle.

An equilateral triangle being inscribed in a circle, the side is $= r\sqrt{3}$ (by Prop. XLI. B. IV. Geometry), and (by Cor. 2.) the perpendicular $= \frac{1}{2}r$, and the sides being 3, the area $= 3 \times \frac{1}{2}rr\sqrt{3} = \frac{3}{2}rr\sqrt{3}$. And since the perp. to the side of the

circum-

circumscribing triangle is twice as much being $= r$, Fig. therefore the area of the circumscribing circle is 4 times as great $= 3rr\sqrt{3}$; and this is to contain one tree.

The area of the circumscribing square, it is evident, is $= 4rr$, which also contains a tree.

In the hexagon, the side of the inscribed figure being r , the perpendicular is $\sqrt{rr - \frac{1}{4}rr} = \frac{r}{2}\sqrt{3}$, and its area $= 6 \times \frac{rr}{4}\sqrt{3} = \frac{3rr}{2}\sqrt{3}$.

But the perpendicular on a side of the circumscribing hexagon being r ; it will be as $\frac{1}{2}rr : rr ::$ area

inscribed $\frac{3rr}{2}\sqrt{3} : 2rr\sqrt{3}$, for the area of the circumscribing hexagon, which likewise contains a tree. Hence the areas of the triangle, square, and hexagon, (that each may contain a tree) are as $3\sqrt{3}$, 4, and $2\sqrt{3}$; and these are as the numbers 3, $2\frac{1}{2}$ and 2 nearly. Hence the hexagons are most advantageous, as taking up less ground, the triangles least proper.

Hence for every 3 trees planted in hexagons, there will but be 2 planted in triangles, in a given quantity of ground. And for every 7 planted in hexagons, there will but be about 6 planted in squares, in the same ground.

Cor. By the above reasoning it appears, that if the trees be placed at the angular points of the figures, the advantage lies the contrary way; triangles the best, squares the next, and hexagons the least; and in proportion as above. For a hexagon is composed of equilateral triangles, whose angular points meet in the center of it. And the angular points of hexagons may be considered as the centers of triangles; the angular points of which triangles meet in the centers of the hexagons.

F f 3

And

Fig. And perhaps this is an easier way in the practice of planting.

The same rules may be applied to any other things, which are to be placed at equal distances.

PROB. X.

To place a number of small spherical bodies, in the least solid space.

94. 1. If several of these spheres be so placed, that all their centers may be in the same plane, and all touch one another; then if the centers of any three touching spheres, be joined by right lines, they will form an equilateral triangle as *abc*. Now if upon this bed, or layer of bodies, another like layer be placed, so that the spheres may fall into the alternate spaces of the former layer, and so on for more layers; then it is evident, that every 4 spheres that touch one another, will have their centers at the 4 angles of a tetrahedron; and this is the constitution of a system of spheres, disposed after this manner.
95. 2. If the spheres are so disposed in squares, so that any 4 touching spheres, having their centers joined by right lines, they will form a square *abcd*. And if upon this layer or bed another such layer be placed, so that the spheres may lie in the hollows or spaces of the former, and so on for other layers. Then it will appear, that any 6 spheres touching one another (and lying in 3 beds or layers) will have their centers at the 6 angles of the octaedron; or, which is the same thing, any 5 touching spheres (in 2 layers) will have their centers in the 5 angles of a half octaedron. And this is the constitution of a system of spheres, disposed after this manner.
3. To find the solid space taken up by any number of these spheres, we must first find the area of the

the base for containing two rows. When the length of the space is given, the number of spheres in any row is the same in both systems; but the breadth of two rows will differ in the two systems. The breadth of the space in the first system = height of the equilateral triangle = $r\sqrt{3}$, (r being the radius of a sphere); and the breadth of two rows in the second system = $2r$, as appears by the figures; this is reckoning from the centers; but reckoning from the outsides of two rows, the breadth = $2r + r\sqrt{3}$ in the first system, and breadth = $4r$ in the second system.

Again for the height, the distance of the center of a sphere, in a row of the second layer = height of the regular pyramid = $2r\sqrt{\frac{2}{3}}$ in the first system; and the distance in the second system = height of half the octaedron = $r\sqrt{2}$; but this is also reckoning from the centers. Therefore reckoning from the outsides, the height of two rows, in the first system = $2r + 2r\sqrt{\frac{2}{3}}$, and in the second system height = $2r + r\sqrt{2}$.

4. Hence, taking two rows in each layer; and the 4 rows will be included in a space in form of a parallelopipedon, whose length in both systems is the same, but the base in the first system = $r \times 2 + \sqrt{3} \times r \times 2 + 2\sqrt{\frac{2}{3}}$, and in the second system = $4r \times r \times 2 + \sqrt{2}$; therefore the spaces to contain the same number of spheres in the first and second systems, are as $13.557rr$ to $rr \times 13.655$; therefore these spaces are as 137 to 138; being nearly equal, but the first system has rather the advantage.

5. In equal spaces the density by the pyramidal method (Fig. 94.) is to the density by the octagonal way (Fig. 95.) is as 138 to 137. For the density is reciprocally as the space taken up by a given quantity of matter.

F f 4

6. If

Fig. 6. If the spheres are so disposed that 8 of them
 94. having their centers connected with right lines,
 95. these lines will form a cube; then the base or section of such a system will be $4r \times 4r$ or $16rr$, which give a space greater than the former, in the ratio of 16 to 13.557 or 13.656, for the same number of spheres.

7. Density of solid space : density syst. art. 6 : : cube of the diameter : to the sphere, that is, as $8a^3 : \frac{4}{3}a^3 \times 3.1416$.

And dens. system art. 6 : density pyram. system : : 13.557 : 16, therefore density of solid space : density of the pyramidal system : : $8a^3 \times 13.557 : 16 \times \frac{4}{3}a^3 \times 3.1416 : : \frac{3}{8} \times 13.557 : 3.1416 : : 5.084 : 3.1416$.

8. If a great heap of small spheres be shaken together in a box; they will dispose of themselves, according to one of the two first systems, and most probably according to the first or pyramidal one, where all their centers are at the angles of regular pyramids.

9. If the spheres are so disposed, that each sphere in the upper layer lies between 2 spheres in the under one, so that all the three centers may be in a perp. plane, and make an equilateral triangle; then the breadth $= 2r$, and height $= r\sqrt{3}$; and whole breadth $= 4r$, and whole height $= 2r + r\sqrt{3}$. Therefore the base $= 4r \times 2r + r\sqrt{3} = 8rr + 4rr\sqrt{3} = rr \times 8 + 6.928 = rr \times 14.928$; which is a greater space than either of the first.

10. Hence the densities in the 4 systems art. 1, 2, 9, 6; are respectively as $\frac{1}{13.557}$, $\frac{1}{13.656}$, $\frac{1}{14.928}$, and $\frac{1}{16}$, and of solid space $\frac{1}{\frac{3}{8} \times 3.1416} = \frac{1}{8.3776}$ by Art. 7.

P R O B.

PROB. XI.

ASB is a compound pendulum, made of the two 96.
bodies A and B, moving about the center of motion
S, and vibrating in a second. To place them so that
the contraction or expansion of the rods Sa, Sb, (made
of different metals) shall cause no difference in the time
of vibration; if it be possible.

It is plain, the bodies A, B, must be on different
sides of S; for if they be on one side; when
one rod is expanded the other will be expanded,
and the pendulum becomes longer, and the time
of vibration longer. And the contrary when they
both contract.

Let d = distance of the center of Oscillation
from S = $39.2 a = SA$, $b = SB$,

Then by Mechanics, $\frac{Aaa + Bbb}{Aa - Bb} = d$. Now

let a expand to a' , and b to b' , both increasing
their length; then to find whether $\frac{Aa'd' + Bb'b'}{Aa' - Bb'}$

be still $= d$, put $A = pB$, p being a fixt number;
then we have, by the first equation, $\frac{pBaa + Bbb}{pBa - Bb}$

$= d$, or $\frac{paa + bb}{pa - b} = d$, now since both a and b are

increased, the whole $paa + bb$ is greater than be-
fore, being increased as the square of a and b .
But $pa - b$ cannot be increased in the same propor-
tion, either of them being only increased as a or b .

Let $qa = b$, then $\frac{paa + qqaa}{pa - qa} = d$, $\frac{p + qq}{p - q} a =$

$\frac{Aaa + Bbb}{Aa - Bb}$, and $\frac{p + qq}{p - q} a'$ is greater than d , since

p and q are fixt numbers. And the same thing
will

Fig. 96. will follow in contractions, for then $\frac{p+qq}{p-q} a'$ will be less than d .

Hence it is impossible to make a compound pendulum of two bodies, and with two rods of any different metals, so as to keep time truly, when they expand or contract.

Otherwise thus.

Since $\frac{paa + bb}{pa - b} = d$, let the expansion of a and b , be x and y ; then $\frac{p \times \overline{a+x^2} + \overline{b+y^2}}{p \times \overline{a+x} - \overline{b-y}} = d$, if it can be done; that is, $\frac{paa + 2pax + bb + 2by \&c.}{pa + px - b - y} = d$, but since $\frac{paa + bb}{pa - b} = d$, therefore the remainder $\frac{2pax + 2by}{px - y} = d$, let $y = nx$, then $\frac{2pax + 2nbx}{px - nx} = \frac{2pa + 2nb}{p - n} = d$, or $\frac{paa + bb}{pa - b} = \frac{2pa + 2nb}{p - n}$, but no numbers can be found that will make them equal, for the latter will always be greater. For if $b=a$, then will $\frac{p+n}{p-n} \times 2a = d$, which is greater than $\frac{paa + aa}{pa - a}$, and if b be increased n will be increased, and p diminished; yet this quantity will always be the greater.

PROB.

PROB. XII.

DF is a water barometer, wherein the water stands 97. at the height AO in the tube at the surface of the earth; then suppose it carried to the top of a hill, where the water rises to B; to find the height of the hill.

Suppose the vessel DF and pipe BA, to be cylinders; the diameter of the vessel $DF = D$, diameter of the pipe $= d$, and $p = ED$ the height of the internal air, $n = AO$ the height of the water in the pipe, $x = OB$ the ascent of the water, all in inches. And $f = 34$ feet the height of a column of water of equal weight with the atmosphere; then $850f =$ height of a uniform atmosphere, $y =$ altitude of the hill.

Suppose, in going to the top of the hill, it rises from O to B in the pipe, and falls from E to C in the vessel. Then the cylinders OB and CF are equal, therefore $EC \times DD = xdd + EC \times dd$, and

$$EC = \frac{xdd}{DD - dd} = xq. \text{ Now OB or } x \text{ would be}$$

the height of the hill, if air was as heavy as water,

therefore $850x =$ its height in air, and $\frac{850x}{1.2} =$

height in air; that is $y = 71x$ in feet. But this is on supposition, that the pressure at C is as great as at E, by which it is raised through the height x ; but the pressure at C being less; the height must be increased in the ratio of the pressures at first and last, that is as DC to DE, and moreover increased by the addition of EC, by which the pipe of water is increased. Whence the height $=$

$$\frac{DC}{DE} + EC \times 71x = \frac{DE + EC}{DE} + EC \times 71x =$$

1 +

Fig. 97. $1 + \frac{qx}{p} + qx \times 71x = 1 + \frac{p+1}{p} qx \times 71x$; that

is $y = 1 + \frac{p+1}{p} \times \frac{ddx}{DD-dd} \times 71x$, the true height in feet.

Cor. If D is very large, the quantity qx may be neglected, and then $y = 71x$, very near.

PROB. XIII.

To find to what distance a fowling piece will kill; having the distance to which one of a given length will kill.

Let f = force of the powder, s = space described, v = velocity generated, t = time, b = body moved, or the shot. Then (Prob. I. Sect. III.

Fluxions,) $\dot{v}\dot{v}$ is as $\frac{fs}{b}$. Then,

1. Suppose f the force of the powder within the barrel to be uniform; then the velocity will be uniformly accelerated to the mouth of the barrel; and f and b being given, $\frac{vv}{2}$ will be as $\frac{fs}{b}$, and v

as $\sqrt{\frac{fs}{b}}$. Therefore put $V = v$, $P = f$, $L = s$, the length of the barrel; and $S = b$ the charge of shot, and we have V as $\sqrt{\frac{PL}{S}}$. And therefore P , S , being given, the velocity at the muzzle will be as the square root of the length.

Let x = distance the shot flies, V = velocity at the gun's mouth, v = killing velocity. Then (Prob. XIII.

Sect. III. Fluxions,) $\frac{3px}{8dg} = 2.3025 \log. \frac{V}{v}$, and

$3px$

$$\frac{3px}{2.3025 \times 8dq} = \log. \frac{V}{v}; \text{ whence } x = \frac{8dq \times 2.3025}{3p} \text{ Fig.}$$

$\log. \frac{V}{v}$, where d is the diam. of a pellet of shot,

q = density of lead, = 10000, p = density of air

= 1. Therefore $x = 61402d \times \log. \frac{V}{v}$, or $x =$

$5117d \log. \frac{V}{v}$ in feet, d being inches. Therefore

V and v being known x is had.

Since V the velocity at the muzzle is found to

be as $\sqrt{\frac{PL}{S}}$; let $V = a \sqrt{\frac{PL}{S}}$; then $x = 5117d$

$\log. a \sqrt{\frac{PL}{S}}$, or $x = 2558d \log. aa \times \frac{PL}{S}$, put-

ting $v = 1$. Then put $P, L, S = 1$, for a gun

that can kill $f(x)$ feet; to be found by experience.

Then $f = 5117d \times \log. a$, and a = number of the

logarithm $\frac{f}{5117d}$. Then a being known, x will be

known for any values of P, L, S .

Note, if one sort of powder be stronger than an-

other, P must be increased in that ratio. *Note also*,

the different diameters of the bores of guns, make

no difference as to the killing distance.

EXAMPLE 1.

If $f = 90$, $d = \frac{1}{12}$, then $\log. a = .211$, and $a =$

1.63 ; whence $x = 426 \times \log. 1.63 \times \sqrt{\frac{PL}{S}}$, for

any values of P, L, S .

EXAMPLE 2.

Let $d = \frac{1}{4}$, $f = 90$, then $5117d = 640$, and

$\log.$

Fig. log. $a = .1407$, and $a = 1.3825$, and $x = 640$

$$\log. 1.382 \sqrt{\frac{PL}{S}}.$$

2. If the force of the powder decreases to a certain distance, so as to be as the distance from the extremity. Put l = length of that space or distance, y = distance from the breach, then the force will be as $l - y$. Then $v\dot{v}$ is as $\left(\frac{F\dot{s}}{b}\right)$ or

$$\frac{l-y}{b} \dot{y} = \frac{l\dot{y} - y\dot{y}}{b}, \text{ and } v^2 \text{ as } \frac{ly - \frac{1}{2}yy}{b}, \text{ whence } v$$

is as $\sqrt{\frac{2ly - yy}{b}}$; and when $y = l$, then v is as

$\frac{l}{\sqrt{b}}$, when l is the length of the bore; but if g = length of the bore, then the velocity at the muzzle

will be $\sqrt{\frac{2lg - gg}{b}}$, or $\sqrt{2lg - gg}$, when b is given.

Having got $V = \sqrt{2lg - gg}$ the velocity at the muzzle, then $x = 5117d \times \log. \frac{V}{v}$ as before, which determines the distance to kill, upon this law of force.

3. If the force of the powder within the gun, be reciprocally as the expansion or space, which at firing is $= c$, the space taken up by the powder; let y = distance from the breach as before, then $v\dot{v}$ is as $\frac{\dot{y}}{y}$, and v^2 as $\log. y$, and corrected v^2 is as $\log.$

$\frac{y}{c}$, and v as $\sqrt{\log. \frac{y}{c}}$. Whence the distance x will be had as before.

But

But as to the force, which of these laws obtains, Fig. is uncertain; and different suppositions, give different conclusions. And this is a matter that must be determined by experiments. But the law being known, every thing else becomes known.

S C H O L.

To find the strength of a gun barrel in every point. The strength must be as the force, and time of acting. In the first case before laid down, the force is uniform, and therefore the strength is as the time of acting, and that strength is as the thickness of the metal. The time of describing the whole length of the barrel, to the time of describing the length y is as $\sqrt{l}$ to $\sqrt{y}$. Therefore the time of acting on a point at the distance y from the breach is $\sqrt{l} - \sqrt{y}$; therefore the thickness of the metal at the distance y from the breach is as $\sqrt{l} - \sqrt{y}$; upon the supposition of uniform force. And the force will be nearly so, if the powder don't fire all at once, but gradually.

P R O B. XIV.

DCFB are two tubes or cylindric vessels that communicate; DC is close at top, FB open at top. Water stands in both at the level CR, and DC contains inclosed air of the same density as the external air. If the tube BF be filled with water as high as B, to find the space BO it will descend in BR. 98.

When the quantity of water BR is put into the tube BR, it will rise to E in the tube DC. To find BO or EC. Let $BO = x$, $EC = y$, $EA = D$, $AF = d$. Then the cylinder QO = cylinder CE.

Whence $DDy = ddx$, and $y = \frac{ddx}{DD} = qx$. Let the horizontal

Fig. horizontal line EF be drawn. Let f = height of
 98. water 34 feet, the weight of the atmosphere, $p =$
 DC , $QR = b$.

Then the pressure at EA = pressure at AF, to
 preserve an equilibrium. The pressure on EA =
 $\frac{DC}{DE} f$, by the compressing force of the air. The

pressure on AF = $f + AO$; therefore $\frac{DC}{DE} f$, or
 $\frac{DE + EC}{DE} f = f + AO$, or $f + \frac{EC}{DE} f = f + AO$,

and $\frac{EC}{DE} f = AO$, and $EC \times f = DE \times AO$, or

$xf = p - y \times b - x - y$, and $\frac{fy}{p - y} = b - x - y$,

and $x = b - y - \frac{fy}{p - y}$, put $q = \frac{dd}{DD}$, $r = 1 + q$;

then the equation reduced is $\frac{fq + bq + pr.x - qrx}{x} = pb$, and $y = qx$.

Cor. In small heights, $x = \frac{pb}{fq}$, nearly.

P R O B. XV.

If a common man 2 yards high can just bear 26lb.
 at arm's end, and the weight of his arm, there be 4lb.
 to find the size of that man that can bear the greatest
 weight possible at arm's end.

Let d = whole weight at arm's end = 30lb. and
 the arm being = half the length of the body = a ,
 the body = $2a$, x = length of the strong man's
 arm, y = the additional weight it can bear at arm's
 end, $b = 4lb$.

The

The weights of similar solids, are as the cubes Fig. 98.
 of the lengths, therefore $a^3 : x^3 :: b : \frac{bx^3}{a^3} = \text{weight}$

of the strong man's arm, at arm's end; to which
 add the additional weight y , which is to be a maxi-
 mum; then $\frac{bx^3}{a^3} + y = \text{total weight at arm's end.}$

The stress on the muscles at the shoulder, is as
 the weight and length of the arm (or lever);
 divided by the distance of the fulcrum; therefore
 the stress is as the weight; because the length of
 the arm, and distance of the fulcrum, is in a
 given ratio.

And the strength of the muscles is but as the
 square of the diameter; or as aa and xx . But the
 stress is as the strength, therefore $aa : xx :: d :$
 $\frac{bx^3}{a^3} + y$, for the common man, and the strong

man; therefore $\frac{bx^3}{a^3} + aay = dxx$, and $aay = dxx -$

$\frac{bx^3}{a^3} = \text{max.}$ Whence $2dxx - \frac{3bx^2x}{a} = 0$, and

$3bx = 2ad$, and $x = \frac{2ad}{3b} = \frac{2 \times 30}{3 \times 4} a = \frac{60a}{12}$

$= 5a$; therefore the strongest man will be 5 times
 higher than the common man.

S C H O L.

In Prob. 187, Algebra, I have computed the
 height of a man that can lift nothing, to be $\frac{d}{b}$
 $a = 7\frac{1}{2}a$. Therefore the height of the common
 man, the strong man, and the great man, will be
 a , $5a$, and $7\frac{1}{2}a$.

PROB. XVI.

Two animals of the same kind being given, a greater and a lesser; to find the proportion of the velocities they can move with, in jumping, running, &c.

Let M, m , be the matter or weight of the two bodies; L, l , their lengths or heights; V, v , the velocities of motion.

By Mechanics (see Prob. I. Sect. III. Fluxions.) vv is as $\frac{fs}{b}$, and vv as $\frac{fs}{b}$. The force (f or) strength of the muscles, is as the squares of their diameters, or as LL to ll . And (s or) the contraction of the muscles, is as their lengths, or as L to l ; and (b or) the weight of the bodies are as M to m , that is as L^3 to l^3 . Therefore $V^2 : v^2 :: \frac{LL \times L}{L^3} : \frac{ll \times l}{l^3}$, that is as 1 to 1; therefore $V = v$.

And the velocities generated are the same in both, by this law of motion, expressed by $vv \propto \frac{fs}{b}$.

Therefore the different size of bodies make no difference in the degree of motion they are capable of performing.

PROB. XVII.

If two weights a and b , be suspended at a rope going over a pulley. To find the space through which the greater body a descends, or the lesser b ascends, in a second of time.

By Schol. Prop. VI. Mechanics, we find $s \propto \frac{Ft^2}{b}$, for uniformly accelerated motion. Let $b =$

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16.1 the feet described in a second by one body Fig. falling freely, as a . $F = \text{weight of that body} = a$, 99.
 $s = b$. Also in the two bodies $x = \text{space described}$
in a second, $F = a - b$, and $b = a + b$, therefore
we have $b : \frac{att}{a} :: x : \frac{a-b}{a+b} tt$, whence $x =$

$\frac{a-b}{a+b} b$, t being the same in both.

EXAM.

If $a = 2b$, then $x = \frac{2b-b}{2b+b} b = \frac{1}{3}b$.

If $a = 3b$, then $x = \frac{3b-b}{3b+b} b = \frac{2}{4}b = \frac{1}{2}b$.

If $na = mb$, then $x = \frac{m-n}{m+n} b$.

SCHOL.

If it was required to find the time of describing any given space S . Having found the space x described in 1 second, it will be $x : 1^2 :: S : \text{Square of the time} = \frac{S}{x}$, and the time $= \sqrt{\frac{S}{x}} =$

$$\sqrt{\frac{S \cdot a + b}{b \cdot a - b}}$$

PROB. XVIII.

In the fire engine, given the weight or strength of the atmosphere upon the piston, and the length of the stroke; to find the water to be drawn at a stroke; so that the greatest quantity shall be drawn in a given time; supposing the force uniform.

Let $a = \text{weight of the atmosphere}$, $y = \text{weight of the water}$, $AB = BC$, $t = \text{time of C's descending to gain a new stroke}$.

Then the force by which the water is drawn up is $a - y$, and the weight $= y$; and (by Schol.

G g 2

Prop.

Fig. Prop. VI. Mechanics), the time of raising y thro

100. a given height will be $\sqrt{\frac{\text{body}}{\text{force}}} (t \propto \sqrt{\frac{bs}{p}})$, that

is as $\sqrt{\frac{y}{a-y}}$; then $t + \sqrt{\frac{y}{a-y}}$ = time of rais

ing y , or the whole time of making a stroke

Then to find how much will be drawn up in the

given time 1, $t + \sqrt{\frac{y}{a-y}}$ (time) : y (weight) ::

(time) : $\frac{y}{t + \sqrt{\frac{y}{a-y}}}$ the weight required = max

put $x = \sqrt{\frac{y}{a-y}}$, then $xx = \frac{y}{a-y}$, and $2xx =$

$\frac{ay}{a-y^2}$, and $\dot{x} = \frac{a\dot{y}}{a-y^2 \times 2\sqrt{\frac{y}{a-y}}}$. Whence

$\frac{y}{t+x} = m$, and $\frac{t\dot{y} + x\dot{y} - \dot{x}y}{t+x^2} = 0$, and $t\dot{y} + x$

$-y\dot{x} = 0$, that is, $t\dot{y} + \dot{y} \sqrt{\frac{y}{a-y}} = y\dot{x} =$

$\frac{a\dot{y}y}{a-y^2 \times 2\sqrt{\frac{y}{a-y}}}$ = $\frac{a\dot{y}y}{2\sqrt{y} \times a-y^3}$, whence

$2t\sqrt{y} \times a-y^3 + 2y \times a-y = ay$, and $2t\sqrt{y} \times a-y$

$= 2yy - ay$, and $4tt \times a-y^3 = 2y - a^2 \times y$; and

the root of this cubic equation will be the value

y for the weight.

If we consider only the time of ascent, then t

set aside, and then $y \sqrt{\frac{a-y}{y}} = \text{max.}$ and $\frac{ay^2}{y}$

or $ay - yy = m^2$, and $a\dot{y} - 2y\dot{y} = 0$, and $a = 2$

or $y = \frac{1}{2}a$.

PROB. XIX.

A pendulous body or door BE, whose weight is 101. given; being shot at with a bullet, whose weight is given, and the arch the door describes after the stroke, being also given; to find the velocity of the bullet.

Let weight of the pendulum $BD = w = 56\frac{3}{8} \text{ lb.}$

weight of the bullet $= m = \frac{1}{12} \text{ lb.}$

the cord of the arch described by E $= p = 17\frac{1}{4} \text{ inches.}$

dist. center of gravity $BD = b = 52 \text{ inch.}$

dist. center of gyration $BR = r = 57.08 = \sqrt{bc.}$

dist. center of oscillation $BA = c = 62\frac{2}{3}.$

dist. center of the pendulum $BC = d = 66.$

velocity bullet impinging at C $= x.$

$BE = g = 71\frac{1}{8}, f = 16.1 \text{ feet.}$

$g : p :: c : \frac{cp}{g} = \text{cord of the arch described by}$

the center of oscillation A, and $\frac{cp}{12 \times 28g} = \text{versed}$

sine of that arch in feet; and $\frac{p}{g} \sqrt{\frac{2fc}{12}} = \text{velocity of A, by falling through that arch} = v.$

By the motion of the pendulum, $\frac{rv}{c} = \text{velocity}$

of the point R. And (Cor. 5. Prop. 59. Mecha-

tics), $\frac{wrv}{c} = \text{momentum of the pendulum at R.}$

And $mx = \text{momentum of the bullet against C,}$

where it impinges. And (Cor. 5. ib.) $\frac{dmx}{r} =$

force of the bullet against R, to give the same mo-

tion to the pendulum; therefore $\frac{dmx}{r} = \frac{wrv}{c}, \text{ and}$

G g 3

$x =$

Fig. $x = \frac{wrrv}{dmc} = \frac{wbcrv}{dmc} = \frac{wbv}{dm} = \frac{pwb}{dmg} \sqrt{\frac{2fc}{12}} =$
 101.

1672 feet in a second.

Note, I suppose the bullet to be at rest after the stroke, tho' it really moves along with the door or pendulum. But this makes but a trifling difference.

S C H O L.

To find the weight of a pendulum placed in C, on which the bullet impinging, shall cause it to describe the same angle about B.

Let Z = weight sought, then $c : d :: v : v \times \frac{d}{c}$
 = velocity of the body Z , at the point C .

And the momentum of Z at C , is $= Zv \times \frac{d}{c}$
 $= mx = \frac{wbv}{d}$ the momentum of the bullet. There-
 fore $Z = \frac{wbc}{dd} = 42\frac{1}{3} lb.$

P R O B. XX.

102. *To find the center of gravity of the logarithmic space ABC; AB being parallel to the asymptote FE.*

Let $GF = 1$, $AD = a$, $FD = b$, DE or $AB = x$, $BC = y$, t = subtangent; then by nature of the curve, $a + y \times \dot{x} = t\dot{y}$, and $y\dot{x} = t\dot{y} - a\dot{x}$, and fl. $y\dot{x} = ty - ax = \text{area } ABC$.

But $y\dot{x}\dot{x}$ = fluxion of the momentum of ABC
 $= t\dot{x}\dot{y} - a\dot{x}\dot{x}$, and fl. $y\dot{x}\dot{x}$ or fl. $t\dot{x}\dot{y} - a\dot{x}\dot{x} = t\dot{x}y - p$
 $- \frac{ax^2}{2}$. This in fluxions is $t\dot{x}\dot{y} + t\dot{y}\dot{x} - \dot{p} - ax\dot{x}$
 $= t\dot{x}\dot{y} - ax\dot{x}$, therefore $\dot{p} = t\dot{y}\dot{x} = t\dot{t}\dot{y} - ta\dot{x}$,
 whence $p = tty - tax$. Therefore fl. $y\dot{x}\dot{x} = t\dot{x}y -$
 tty

$ty + tax - \frac{ax^2}{2} =$ momentum of ABC. Whence Fig. 102.

d , the distance of the center of gravity from A, is

$$= \frac{txy - tly + tax - \frac{1}{2}ax^2}{ty - ax}, \text{ or } d = \frac{ty \times x - t + t - \frac{1}{2}x \times ax}{ty - ax}.$$

Cor. 1. When $a=1$, $d = \frac{ty \times x - t + t - \frac{1}{2}x \times x}{ty - x}$.

Cor. 2. The distance of the center of gravity from BC, is $= t - \frac{\frac{1}{2}axx}{ty - ax}$.

For $x - d = x - \frac{ty \times x - t + t - \frac{1}{2}x \times ax}{ty - ax} =$

$$\frac{tly - tax - \frac{1}{2}axx}{ty - ax} = t - \frac{\frac{1}{2}axx}{ty - ax}.$$

Again, to find the distance from the axis of suspension AB. The fluxion of the momentum $= \frac{1}{2}y \times y\dot{x} = \frac{1}{2}y \times ty - ax = \frac{1}{2}ty\dot{y} - \frac{1}{2}ay\dot{x} = \frac{1}{2}ty\dot{y} - \frac{1}{2}a \times ty - ax = \frac{1}{2}ty\dot{y} - \frac{1}{2}aty + \frac{1}{2}aax$. And the fluent $= \frac{ty^2}{4} - \frac{aty}{2} + \frac{aax}{2}$, the momentum; therefore $\frac{ty^2 - 2aty + 2aax}{4 \times ty - ax} =$ distance of the center of gravity from AB.

PROB. XXI.

To find the nature of the curve ADE, which lying with its plane parallel to the horizon, and supposed to be of a given thickness q ; and being fixt with its end in the wall ER; it shall be equally strong every where to support its own weight. 103.

The meaning of the question is this, let C be the center

G g 4

Fig. center of gravity of the area ABD, in the axis 103. AB. Then the area $BAD \times BC = BD \times qq$.

Let $AB = x$, $BD = y$, then area $BAD = fl. y\dot{x}$. And by the nature of the center of gravity, $AC = \frac{fl. yx\dot{x}}{fl. y\dot{x}}$, the point of suspension being at A. There-

fore $BC = x - \frac{fl. yx\dot{x}}{fl. y\dot{x}}$, and therefore $BAD \times BC$

or $fl. y\dot{x} \times x - \frac{fl. yx\dot{x}}{fl. y\dot{x}} = y \times qq$; that is, $x fl. y\dot{x}$

$- fl. yx\dot{x} = qqy$. This in fluxions is $\dot{x} fl. y\dot{x} + yx\dot{x} - yx\dot{x} = qq\dot{y}$; that is, $\dot{x} fl. y\dot{x} = qq\dot{y}$. This put into fluxions making $\dot{x}$ constant, will be $y\ddot{x}\dot{x} = qq\ddot{y}$; multiply by $\dot{y}$, then $y\dot{y}\dot{x}^2 = qq\dot{y}\dot{y}$, and the fluent is $y^2\dot{x}^2 = qq\dot{y}^2$, and $y\dot{x} = q\dot{y}$, which is the property of the logarithmic curve, whose subtangent is q .

But it is not any portion of the logarithmic curve which will answer the question, but only a part at the very small end, where the curve is indefinitely near the asymptote, which suppose to begin at A; the weight of the rest beyond, being supposed of no consequence. But by Cor. 2. of the last Prob. when a is inconsiderable, the distance of the center of gravity from B, that is, $BC = \left(t - \frac{\frac{1}{2}axx}{ty - ax}\right) t$, and the area $ABD = ty$; therefore $t \times ty$, which is the stress at B, is as qqy , which is the strength at B; that is, they are in a given ratio as required. And this is all the solution the question is capable of.

PROB. XXII.

To construct the solid of least resistance, for the figure of a ship that will sail the fastest possible.

Let $x =$ abscissa, $y =$ ordinate, and $z =$ curve; then (by Ex. 17. Prob. I. Sect. II. Fluxions), the equation

equation of the curve generating (by its revolution Fig. about it axis) the solid of least resistance is $yy^3\dot{x} = 103$.

$a\dot{x}^4$, that is, $yy^3\dot{x} = a \times \dot{x}^4 + 2\dot{x}^3\dot{y} + \dot{y}^4$. Here x one of the variable quantities is wanting. Therefore by Rule VI. Sect. I. Fluxions, (see Ex. 39.)

assume $\frac{s\dot{y}}{a} = \dot{x}$, and expunging x , we have $aasy$

$$= s^4 + 2aass + a^4. \text{ Whence } y = \frac{s^3}{aa} + 2s + \frac{aa}{s},$$

$$\text{and } \dot{y} = \frac{3s\dot{s}}{aa} + 2\dot{s} - \frac{aas}{ss}, \text{ therefore } \frac{s\dot{y}}{a} \text{ or } \dot{x} =$$

$$\frac{3s^3\dot{s}}{a^3} + \frac{2s\dot{s}}{a} - \frac{as}{s}, \text{ Whence the fluent is } x =$$

$$\frac{3s^4}{4a^3} + \frac{ss}{a} - a \times 2.30258 \log. s. \text{ Therefore } y \text{ be-}$$

ing known s will be known (by the equation $y =$

$$\frac{s^3}{aa} + 2s + \frac{aa}{s}), \text{ and consequently } x \text{ by the last equation.}$$

Now for constructing the curve, suppose $x = 30$ feet, the length of the midship, and $y = 10$ feet, half the greatest breadth of the ship; then from the two equations,

$$\frac{3s^4}{4a^3} + \frac{ss}{a} - a \times 2.3025 \log. s = 30,$$

$$\text{and } \frac{s^3}{aa} + 2s + \frac{aa}{s} = 10,$$

we shall find $s = .519543$, and $a = .125311$: for

taking s at pleasure, finds $a = \frac{ss}{\sqrt{10s - a}}$. And

by correcting the assumption, a and s are found by degrees, as above. Now the value of a being substituted in the equations above, gives these two following equations,

Fig.

103.

$$1. x = \frac{s^4}{.002623} + \frac{ss}{.1253} - .2885 \log. s.$$

$$2. y = \frac{s^3}{.0157} + 2s + \frac{.0157}{s}.$$

Instead of $.2885 \log. s$, you may take $.1442 \log. ss$. When x or y is a minimum, we shall find $s =$

$$\frac{a}{\sqrt{3}}, \text{ whence } x = \frac{s}{1\frac{1}{2}} a - a \times 2.3025 \log. \frac{a}{\sqrt{3}}.$$

$$\text{And } y = \frac{16a}{3\sqrt{3}}; \text{ or } x = .38125, \text{ and } y = .385858.$$

From the foregoing equations, the values of x , s , and y , will be found as in the following Table; by the help of which numbers, the curve may be easily constructed.

ss	x	y	ss	x	y	ss	x	y
.03630	1	.904	.15883	11	4.869	.22393	21	7.730
.05944	2	1.485	.16643	12	5.179	.22948	22	7.993
.07650	3	1.957	.17372	13	5.483	.23489	23	8.253
.09057	4	2.390	.18071	14	5.780	.24020	24	8.510
.10290	5	2.793	.18747	15	6.072	.24537	25	8.764
.11400	6	3.174	.19401	16	6.360	.25045	26	9.016
.12420	7	3.538	.20033	17	6.641	.25544	27	9.265
.13363	8	3.886	.20643	18	6.917	.26033	28	9.511
.14252	9	4.224	.21244	19	7.191	.26514	29	9.756
.15080	10	4.552	.21827	20	7.463	.26986	30	10.

The process is thus; for the given value of x , find s by the first equation, which would be a quadratic equation, if the logarithmic quantity was wanting. Therefore it will be the easiest way to find s by trial and error. When s is had, y will be found directly.

PROB.

PROB. XXIII.

If AA and BB be the position of the sails of a wind-^{104.} mill, making an angle of $54\frac{1}{2}$ with the axis of the mill XC; the arms of the sails being perp. to the horizon at C. To determine the angle WCX of the winds direction WC, so as to have the greatest force on the sails.

Here we suppose BB to be the top of one sail, and AA the bottom of the other; and that BCA = an angle of twice $54\frac{1}{2}$. The common situation of a windmill is that XC is the direction of the wind; here WC is the direction.

Suppose $r = CA =$ radius; and put $s = AE$ or BE the S. ACE $54\frac{1}{2}$. S. WCE = $x = OX$, CE = b , CO = z . Then (Trig. I. 5.) $\frac{sz + bx}{r} =$

An the S. ACW, and (ib. 6.) $\frac{sz - bx}{r} = BF$.

Then (Mechanics, CII.) $An^2 =$ force on the sail CA, and (ib. Cor. 2.) $An^2 \times b =$ force of the wind on the sail SA, to move it in direction CH. Likewise $BF^2 \times b =$ force on the sail CB to move it in the same direction CH. But the sum of these

is a maximum. That is, $\frac{sszz + 2szbx + bbxx}{rr} b +$

$\frac{sszz - 2szbx + bbxx}{rr} b = \text{max. that is, } \frac{2sszz + 2bbxx}{rr} b$

$= \text{max. and } sszz + bbxx = \text{max. or } ssrr - ssxx + bbxx = \text{max. (because } rr - xx = zz), \text{ therefore } -2ssxx + 2bbxx = 0, \text{ and } bbx = ssx, \text{ whence } x = 0, \text{ and therefore the wind has the greatest force possible upon the sails of a windmill, to turn them round when it blows in the same direction with the axis of the mill XC.}$

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If $AE = CE$, (that is, if the sail makes an angle of 45 with the axis), then the sum of the forces $sszz + bbxx$, will be $sszz + ssxx$, because $b = s$. Therefore $sszz + ssxx = \max.$ and $zz + xx = \max.$ that is, $rr - xx + xx = \max. = rr$, whose fluxion is 0, and rr the sum of the forces is a standing quantity. And therefore the force will be the same in all directions, between AC and BC,

P R O B. XXIV.

Having given the angle of the sails of a windmill with the axis $54^{\circ}\frac{1}{2}$, the velocity of the wind 14 feet in a second, and the time of one revolution of the sails 6 seconds. To find that point of the arm or sail, which is not at all acted on by the wind,

105. Let AB be the top of the sail, draw AD for the way of the wind, perp. to BD the way of the sail, and let $AB = b$, the breadth of the sail. S. DAB, $54^{\circ}\frac{1}{2} = s$, its cosine $= p$, velocity of the wind $= v = 14^f$, and time of the sails revolution $= t = 6''$, $x = CA$, A being the point required, $c = 3.1416$.

Now that the wind may not act on the sail, the point B must move through BD, in the same time the wind moves through AD. For then the same particle of air will slide along AB without acting on it. Now if $AB = b$, $BD = sb$, and $AD = pb$; then $2cx =$ circumference described by A or B in one revolution. And $v : 1'' :: pb (AD) : \frac{pb}{v} =$ time of describing AD by the wind. Also $2cx : t (6'') :: sb (BD) : \frac{tsb}{2cx} =$ time of describing BD by

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the motion. Whence $\frac{pb}{v} = \frac{tsb}{2cx}$, and $2cpbx = tsbv$, 105. Fig.

whence $x = \frac{tsv}{2cp} = 18.74$.

Cor. Hence the sails of a windmill should never be made longer than x ; for if they be, what is beyond will strike the back wind, and retard their motion. Nor should they be made quite so long; for a part near the end will be in a manner useless.

PROB. XXV.

To find the weight of a pillar of marble infinitely high, whose base is B.

Let r = radius of the earth, x = any height above it, and x' any small height, and Bx' = weight of a small part of the cylinder or pillar, at the earth's surface. And by the laws of gravity $\frac{1}{rr}$

: Bx' :: $\frac{1}{r+x^2}$: $\frac{rrBx'}{r+x^2}$, the weight of the part Bx' , at the height x . Therefore the fluxion of the weight is $\frac{rrB\dot{x}}{r+x^2} = rrB\dot{x} \times \overline{r+x}^{-2}$; and the flu-

ent is $-rrB \times \overline{r+x}^{-1}$ or $\frac{-rrB}{r+x}$; and corrected,

the true fluent is $\frac{rrB}{r} - \frac{rrB}{r+x}$, for the weight of the pillar whose height is x . That is, the weight

$= rrB \times \frac{1}{r} - \frac{1}{r+x} = rrB \times \frac{x}{r \times r+x} = rB \times \frac{x}{r+x} \times \text{spec. gravity of marble.}$

Therefore

Fig. Therefore when x is infinite, the weight of the
 105. infinite pillar of marble is $= rB \times \frac{x}{x} = rB \times \text{spec. gravity.}$

Cor. The weight of an infinite pillar of any matter is = weight of the solid hyperboloid, generated by an hyperbola revolving about the asymptote; whose base is B , and inscribed parallelogram is rd , where $d = \text{radius of the base of the pillar, and the gravity undiminished.}$

By Ex. 9. Prob. XIV. Sect. II. Fluxions.

P R O B. XXVI.

To find the strength of a piece of iron and a piece of wood, of the same length, and of equal weight. Also to find the weights, when the strength is the same.

106. 1. We shall suppose these to be two square pieces of equal length. Let A be a beam of iron, B a beam of wood of the same bigness, C a beam of iron of the same weight as B . Let $y = \text{side of the square of } A \text{ and } B, x = \text{that of } C, w = \text{the strength of wood, } n = \text{strength of iron, } s \text{ and } t = \text{specific gravities of wood and iron. Now that } B \text{ and } C \text{ may have the same weight, } sy^3 = txx^3; \text{ and strength of the wood } B : \text{strength iron } A :: w : n; \text{ and strength of iron } A : \text{strength of iron } C :: y^3 : x^3. \text{ Whence strength of wood } B : \text{strength of iron } C :: wy^3 : nx^3 :: \left(\text{because } y = \sqrt{\frac{txx^3}{s}} \right) \frac{wtxx}{s} \sqrt{\frac{txx}{s}} : nx^3 :: \frac{wt}{s} \sqrt{\frac{t}{s}} : n :: wt\sqrt{t} : ns\sqrt{s}. \text{ That is, strength of the wood } B : \text{to the strength of the iron } C, \text{ of the same weight} :: \frac{w}{s\sqrt{s}} : \frac{n}{t\sqrt{t}}.$

Suppose

Suppose elm or ash whose specific gravity is .8, Fig. and iron is 7.6. Also $8\frac{1}{2}$ and 107 are the strength₁₀₆ of this wood and iron. Therefore $w = 8\frac{1}{2}$, $n = 107$, and $s = .8$, $t = 7.6$. Therefore strength of the wood : strength of iron of the same weight; as $\frac{8\frac{1}{2}}{.8\sqrt[3]{.8}}$ to $\frac{107}{7.6\sqrt[3]{7.6}}$ or as 11.8 to 5.1; therefore this wood is twice as strong for the weight as iron.

2. Suppose B and C of the same strength, in which case $wy^3 = nx^3$, and $y = x \sqrt[3]{\frac{n}{w}}$. But weight

of B : weight of C :: $syy : txx :: sxx \sqrt[3]{\frac{nn}{ww}} : txx$

$$:: s \sqrt[3]{\frac{nn}{ww}} : t :: \frac{s}{\sqrt[3]{ww}} : \frac{t}{\sqrt[3]{nn}}.$$

Suppose the wood to be oak, then $s = .8$, $t = 7.6$, and $w = 11$, $n = 107$. Therefore the weight of oak : to the weight of iron of the same strength

:: is as $\frac{.8}{\sqrt[3]{11^3}}$: to $\frac{7.6}{\sqrt[3]{107^3}}$:: that is, as 16 to 34; therefore a piece of iron of the same strength as a piece of oak, weighs as much more.

PROB. XXVII.

Suppose a stream of water to run down the plane AC, whose elevation above the horizon GCH is 30° ; 107. and the perp. BC be 8 feet.

To find the radius of the water wheel DBF, so that the effect of the wheel, driven by this stream of water, shall be a maximum.

Let E be the center of the wheel, and draw EB, ED, perpendicular to AC, CG, and draw CE. Since the angles at D and B are right, the angle DEB = ACH = CAP, supposing APF a horizontal line; and the angle CEB = $\frac{1}{2}$ CAP = 15° .

Let

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Fig. Let $EB = x$, $t = \tan. BEC = \tan. 15^\circ$, $AC = a$, $PC = p$, $S. PAC = S$; then radius $(r) : EB (x) :: \tan. BEC (t) : tx = CB$, and $AB = a - tx$.

Also $AC (a) : CB (p) :: AB (a - tx) : p - \frac{ptx}{a}$

$=$ the depth at B. But the effect of the wheel, will be as the velocity of the water on the floats at B, and as the radius of the wheel EB, or as $x \times$

$\sqrt{p - \frac{ptx}{a}}$; therefore $x \sqrt{p - \frac{ptx}{a}} = \text{max.}$ and xx

$\times a - tx$ or $axx - tx^2 = \text{max.}$ In fluxions $2axx$

$- 3tx^2 = 0$, or $2a = 3tx$, whence $x = \frac{2a}{3t} =$

$\frac{2p}{3s}$, because $a = \frac{p}{s}$, therefore $x = 39.8$ feet.

Cor. If PAC be a right angle, or the water descend perpendicularly down PC ; then $t = \tan. 45^\circ = 1$, $a = PC = p$; then $x = \frac{2}{3}a = \frac{2}{3}PC = 5\frac{1}{3}$ feet.

P R O B. XXVIII.

Given the length of a cylinder CB , and its inclination to the horizon; to find the time of its falling to the ground CE .

108. The time of its falling will be the same as the time of a body's falling through the arch DE of a circle, whose radius $CD = \frac{2}{3}CB$, D being the center of oscillation. Therefore to find the time of descent of D through the arch DE .

Draw DF , gI , perp. and Dh parallel to AC , where AC is perp. to the horizon; and let $CD = r$, $AF = a$, $FC = s$, $Dh = x$, $gI = y$, $DI = z$, $b = r + s$, $t = \text{time of descending through } DI$, $d = 16.1$ feet; then $2d = \text{velocity acquired by a falling body in } t''$. Also $\sqrt{d}$ (height) : $2d$ (velocity) :: $\sqrt{x}$

$\sqrt{x}$ (height) : $\sqrt{2dx}$ = velocity at b or I. Also, Fig. 108.

$2d$ (space) : $1''$ (time) : x' (space) : $\frac{x'}{2d}$ = time of

describing x' . But (Fluxions, Prob. I. Sect. III.)

$i \propto \frac{s}{v}$, and therefore $\frac{x'}{2d}$ (time) : $\frac{x'}{2d}$ ($\frac{\text{space}}{\text{vel.}}$)

: t' (time) : $\frac{z'}{2\sqrt{dx}}$; therefore $i = \frac{z}{2\sqrt{dx}}$. But

$z : x :: r : y$, and $z = \frac{rx}{y}$, and $yy = a + x \times$

$b - x$. Therefore $i = \frac{rx}{2y\sqrt{dx}} =$

$\frac{rx}{2\sqrt{dx}\sqrt{a+x}\sqrt{b-x}} = \frac{r}{2\sqrt{d}} \times \frac{x^{-\frac{1}{2}}}{\sqrt{a+x}\sqrt{b-x}}$

$= \frac{r}{2\sqrt{d}} \times \frac{x^{-\frac{1}{2}}}{\sqrt{ab + 2sx - xx}}$. And the fluent

found by infinite series, will give t .

And if t be given, then by reversion of series, x may be found.

PROB. XXIX.

Supposing a ship without lee way; and WS be the direction of the wind, SA the position of the sail; SD the position of the keel, or ship's way. Any one of the angles WSA, ASD, or WSD, being given; to find the rest, so that the ship may gain the most possible to windward in a given time. 109.

Let the S.WSA = s , its cof. ASG = c , S. ASD = x , cof. WSD or S. DSG = z , supposing SGP perp. to WS; r = radius.

Then (by Cor. 5. Prop. CVI. Mechanics,) the velocity of the ship to windward is as S. WSA $\times \sqrt{S. ASD} \times \text{cof. WSD}$; that is, as $sx^{\frac{1}{2}}z$. Then,

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Fig. 1. Suppose WSA or s to be given; then $\frac{1}{2}z$, or xzz 109, = a maximum. Whence $\dot{x}zz + 2z\dot{z}x = 0$, and $z\dot{x} + 2x\dot{z} = 0$. But (by Prop. VI. B. I. Trigonometry,)

$$\frac{c\sqrt{rr-xx}-sx}{r} = z, \text{ and } \frac{-c\dot{x}\dot{x}}{r\sqrt{rr-xx}} - \frac{s\dot{x}}{r} = \dot{z},$$

$$\text{therefore } z\dot{x} + 2x \times \frac{-c\dot{x}\dot{x}}{r\sqrt{rr-xx}} - \frac{s\dot{x}}{r} = 0,$$

$$\text{and } z = \frac{2c\dot{x}\dot{x}}{r\sqrt{rr-xx}} + \frac{2sx}{r} = \frac{c\sqrt{rr-xx}-sx}{r},$$

$$\text{and } \frac{2c\dot{x}\dot{x}}{\sqrt{rr-xx}} + 3sx = c\sqrt{rr-xx}, \text{ and } 2c\dot{x}\dot{x} +$$

$3sx\sqrt{rr-xx} = crr - c\dot{x}\dot{x}$, or $3sx\sqrt{rr-xx} = crr - 3c\dot{x}\dot{x}$. involved is $9ssrrxx - 9ssx^2 = c^2r^2 - 6ccrrxx + 9ccx^2$, which reduced (putting $rr = ss + cc$), then $9x^4 - 9ssxx - 6ccxx + crr = 0$; which equation contains 4 roots, two affirmative and two negative. And the angle DSG will be about double to ASD, or their fines.

Hence if WSA = 0, 30, 45, 60, 90; then ASD will be = 35, 22, 16, 10, 0, respectively.

2. Suppose x or the angle ASD to be given; then $sz = \max.$ and $s\dot{z} + z\dot{s} = 0$, or $z\dot{s} = -s\dot{z}$.

Let $d = \cos. ASD = \sqrt{rr-xx}$, and $c = \sqrt{rr-ss} = \cos. ASW = S. ASG$. Therefore instead of

$$\frac{c\sqrt{rr-xx}-sx}{r} = z, \text{ we have } \frac{d\sqrt{rr-ss}-sx}{r} = z,$$

$$\text{and } \dot{z} = \frac{-d\dot{s}\dot{s}}{r\sqrt{rr-ss}} - \frac{x\dot{s}}{r} = 0, \text{ whence } z\dot{s} =$$

$$\frac{d\dot{s}\dot{s}}{r\sqrt{rr-ss}} + \frac{x\dot{s}}{r}, \text{ and } z = \frac{dss}{r\sqrt{rr-ss}} + \frac{xs}{r} =$$

$$\frac{d\sqrt{rr-ss}-sx}{r}, \text{ therefore } \frac{dss}{\sqrt{rr-ss}} + 2sx =$$

$$d\sqrt{rr-ss}, \text{ and } dss + 2sx\sqrt{rr-ss} = drr - dss, \text{ and}$$

$2sx\sqrt{rr-ss} = drr - 2dss$, and $4ssrrxx - 4xxs^4 =$ Fig. 109.
 $dd \times rr - 2ss^2$, and putting $rr - xx$ for dd , and reducing, we have $4rrs^4 - 4r^4ss - r^4x^2 + r^6 = 0$,
 and $4s^4 - 4rrss + r^4 = rrx$, and extracting the root, $2ss - rr = rx$, or $rr - 2ss = rx$. But (by Trigonometry) $\frac{rr-2ss}{r}$ = is the cosine of $2WSA$;

and since $x = S.DSA = \cos. WSA + DSG$, therefore $2WSA = WSA + DSG$, therefore $WSA = DSG$; and either of them $= \frac{1}{2}$ comp. of ASD .

3. If WSD or z be given, then put $n = S.WSD$. Then $s\sqrt{x}$ or $ssx = \max.$ and $zssx + ss\dot{x} = 0$, and $2xs = -s\dot{x}$. But (by Trigonometry) $\frac{n\sqrt{rr-ss}-zs}{r}$

$= x$, and $\frac{-nss}{r\sqrt{rr-ss}} - \frac{zs}{r} = \dot{x}$, whence $2xs =$

$\frac{nss}{r\sqrt{rr-ss}} + \frac{zss}{r}$, and $x = \frac{nss}{2r\sqrt{rr-ss}} + \frac{zs}{2r} =$

$\frac{n\sqrt{rr-ss}-zs}{r}$, and $\frac{nss}{2\sqrt{rr-ss}} + 3zs = 2n\sqrt{rr-ss}$

and $nss + 3zs\sqrt{rr-ss} = 2nrr - 2nss$, and $3zs\sqrt{rr-ss} = 2nrr - 3nss$, and $zz \times 9rrss - 9s^4 = 4n^2r^4 - 12n^2r^2ss + 9n^2s^4$, and putting $rr - nn$ for zz , and reducing $9s^4 - 9rrss + 4rnnn = 0$,
 $-3nn$

from which equation s will be found.

If WSD be between 45 and 60 degrees, then WSA is nearly twice ASD .

110. *There is a river FKD runs due east, and several swimmers of equal swiftness, set off from a point A in the north side; and move in all directions, or all degrees of the arch KBD. To find which swimmer will land on the opposite side, the furthest up the river.*

Let b = breadth AD or AB, and let b represent the swimmers motion, and d = BC the currents motion, $AE = x$, $EB = y$. Then the swimmer going off in direction AB, by the compound motion, goes in the line AC, and lands at H, so that HD is a minimum, by similar triangles, $AE (x)$:

$$EC (d - y) :: AD (b) : DH = \frac{b}{x} \times \overline{d - y} =$$

$$\text{min. and } \frac{d - y}{x} = \text{min. whence } \frac{-xy - \overline{d - y} \times \dot{x}}{xx}$$

$$= 0, \text{ and } -xy = \overline{d - y} \times \dot{x}, \text{ but } xx + yy = bb,$$

$$\text{and } xx = -yy, \text{ or } -\dot{y} = \frac{xx}{y}; \text{ therefore } \frac{xxx}{y}$$

$$= dx - y\dot{x}, \text{ and } xx = dy - yy = bb - yy, \text{ therefore } dy = bb, \text{ and } y = \frac{bb}{d}.$$

Cor. 1. *The direction AC will touch the circle GCH, equal to the circle KBD; for all the lines GK, CB, cb, &c. are equal; d being greater than b.*

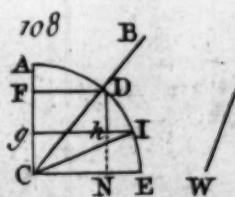
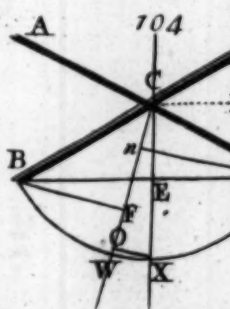
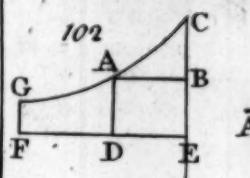
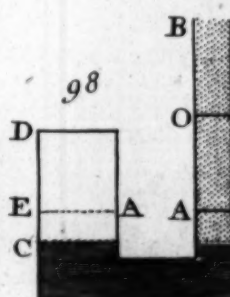
Cor. 2. *If b is equal or greater than d, the swimmers direction is extremely near AK.*

$$\text{Cor. 3. } HD = \sqrt{dd - bb}.$$

Cor. 4. *If the time of passing through AB be = t ;*
the time of arriving at H will be $\frac{dt}{\sqrt{dd - bb}}$.

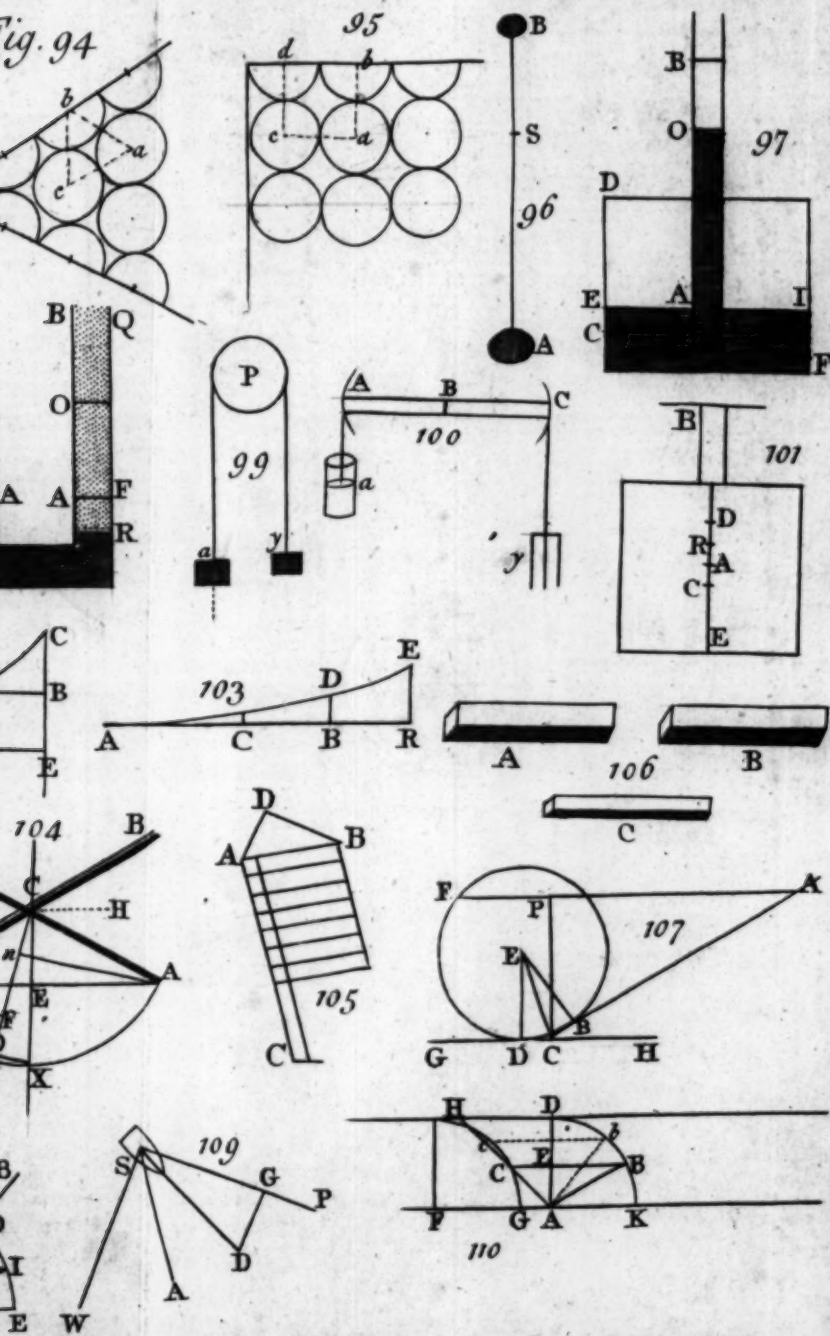
For

Fig. 94



Miscellanies.

Fig. 94



For $x \left(\sqrt{bb - \frac{b^4}{dd}} \right) : t :: b : \text{time sought.}$

Fig.

S C H O L.

The river will be crossed in the least time, by moving in the perpendicular direction AD.

P R O B. XXXI.

Let CD be an exponential curve, whose equation is $y = x^x$, putting $AB = x$, $BD = y$. To find the area, and the least ordinate.

1. Let $X = \text{hyp. log. } x$, $Y = \text{hyp. log. } y$. Then since $y = x^x$, therefore $Y = xX$; and by the nature of logarithms (see Schol. 2. Prob. II. Sect. II.

Fluxions,) $y = 1 + Y + \frac{Y^2}{2} + \frac{Y^3}{6} + \frac{Y^4}{24} \&c.$

$= 1 + xX + \frac{x^2 X^2}{2} + \frac{x^3 X^3}{3}, \&c.$ and the fluxion

of the area $y\dot{x}$ or $x^x \dot{x} = \dot{x} + Xx\dot{x} + \frac{X^2 x^2 \dot{x}}{2} +$

$\frac{X^3 x^3 \dot{x}}{6} + \frac{X^4 x^4 \dot{x}}{24} \&c.$ And the fluent $= x +$

$\frac{x x X}{2} + \frac{x^2 X^2}{6} + \frac{x^3 X^3}{24} \&c. - s.$ And proceed-

ing by Rule 8, Sect. I. Fluxions, $s = \frac{x^2}{4} + \frac{x^3 X}{9}$

$+ \frac{x^4 X^2}{32} \&c. - t.$ And $t = \frac{x^3}{27} + \frac{x^4 X}{43} \&c. - u.$

And $u = \frac{x^4}{4^4}$, and so on; therefore fluent $x^x \dot{x} =$

$x + \frac{x^2 X}{2} + \frac{x^3 X^2}{6} + \frac{x^4 X^3}{24}$ and $-s + t - u \&c.$

Or the area

H h 3

=

Fig.
III.

$$\begin{aligned}
 &= x + \frac{xxX}{2} + \frac{x^3X^2}{6} + \frac{x^4X^3}{24} \&c. \\
 &\quad - \frac{x^2}{4} - \frac{2x^3X}{18} - \frac{x^4X^2}{32} \\
 &\quad \quad + \frac{x^3}{27} + \frac{x^4X}{4^3} \\
 &\quad \quad \quad - \frac{x^4}{4^4} \\
 &= x + \frac{xY}{2} + \frac{xY^2}{6} + \frac{xY^3}{24} \&c. \\
 &\quad - \frac{x^2}{4} - \frac{2x^2Y}{18} - \frac{x^3Y^2}{32} \\
 &\quad \quad + \frac{x^3}{27} + \frac{x^3Y}{4^3} \\
 &\quad \quad \quad - \frac{x^4}{4^4}
 \end{aligned}$$

But when $x = 0$, the area $= 0$, and $X = -$ infinity, but xX , x^2X^2 , x^3X^3 , &c. are each $= 0$. Therefore the area above is correct.

Cor. When $x = 1$, then $X = 0$, and the area $= x$

$$\frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} \&c.$$

2. For the least ordinate, when y or x^x is a minimum, then y or $xX =$ minimum; whence $x\dot{X} + X\dot{x} = 0$, that is, $\frac{x\dot{x}}{x} + X\dot{x} = 0$, and $X = -1$,

the hyp. log. x ; and the common log. $x = \frac{-1}{2.3025} = .434294 = -1.565706$, and $x = .36788$, and $y = .6922$.

P R O B. XXXII.

If two bodies p , q , be suspended by ropes on the axis in peritrochio; to find the pressure on the axis, when the bodies are left at liberty to descend.

Let

Let rad. wheel $Ab = a$, rad. of the axis $CB = b$, Fig. $b = 16.1$, the height descended in a second by gravity; and suppose p outweighs q on the machine, divide the weight p into two parts n and r , so that

CA or $Cb \times r = CB \times q$, or $r = \frac{bq}{a}$, then $p =$

$n + \frac{bq}{a}$; and hence it is plain the weight r acting

at A , and the weight r acting at b (or q acting at B) will keep the machine in equilibrio, and it will remain at rest. Suppose then (instead of q) that the weight $n + r$ hangs at A , and r at b ; the force that gives motion to the wheel is the part n acting at A . Now to find the space x through which these bodies are drawn in a second, by that force.

When the time is given, the space is as the force divided by the body (by Mechanics); therefore

$$\frac{n}{n} \left(\frac{\text{force}}{\text{body}} \right) : b \text{ (space)} :: \frac{n}{p+r} \left(\frac{\text{force}}{\text{body}} \right) : x =$$

$$\frac{nb}{p+r} = \frac{p-r}{p+r} b, \text{ the space through which } r \text{ is}$$

raised upwards. But when the time is given, the space is as the force that generates it, $b \text{ (space)} : r$

$$\text{(force)} :: x \text{ (space)} : \frac{rx}{b} = \text{force acting at } A \text{ and}$$

b , that drives r up. But the force acting at B to

drive q up, is $\frac{a}{b} \times \frac{rx}{b}$; and both these forces

must be supported by the axis at C , or else they

could not act against one another; therefore $\frac{rx}{b} +$

$\frac{arx}{bb}$ is the pressure upon the axis by the motion of the machine; to which we must add the dead weight which the machine supported when it was

Fig. at rest; and this additional weight is $q + r$, because r (suspended at A) balances q (suspended at B). Whence the whole pressure upon the axis is

$$\frac{a+b}{b} \times \frac{rx}{b} + q + r = \frac{a+b}{bb} \times \frac{p-r}{p+r} rb + q + r = \frac{a+b}{b} r \times \frac{p-r}{p+r} + q + r, \text{ where } r = \frac{bq}{a}.$$

E X A M.

Suppose $a = 5$, $b = 2$, $p = 4$, $q = 9$, then $r = 3.6$; and the weight $= \frac{25.2}{2} \times \frac{4-3.6}{4+3.6} + 9 + 3.6 = .66 + 9 + 3.6 = 13.26$.

P R O B. XXXIII.

113. In the log. spiral AfB , if the ordinates CA , CB , be given, and the angle ACB . To find the angle of the spiral, or angle B .

Draw the arch of the circle Apq ; and let there be an indefinite number ($n - 1$) of mean proportionals Cc , Cd , Ce , &c. between AC and BC . And let $AC = a$, $BC = b$, $Cc = x$, arch $Apq = H$; then $a : x : \frac{xx}{a} : \frac{x^3}{aa} : \frac{x^4}{a^3} : \frac{x^5}{a^4} \dots \frac{x^n}{a^{n-1}} = b$; whence Cc is the first of $n - 1$ mean proportionals, between AC and CB : and the arch $Ar = \frac{1}{n}$ of the arch Apq . Then $\frac{x^n}{a^n} = \frac{b}{a}$, and $\log. \frac{x^n}{a^n} = \log. \frac{b}{a} = \log. b - \log. a = B - A$, (putting B , A , for these logs.) that is $n \times \log. \frac{x}{a} = B - A$,

— A, and $\log. \frac{x}{a} = \frac{B-A}{n} = L$; whence $\frac{x}{a}$ Fig. 113.

$$= \text{num. of the log. } L = 1 + \frac{L}{m} + \frac{L^2}{2m^2} +$$

$\frac{L^3}{6m^3}$ &c. by the nature of logarithms: where $m=1$,

for the hyp. logarithms, or $m = .43429$ for the

common logarithms. Whence $x = a + \frac{aL}{m} +$

$\frac{aL^2}{2m^2}$ &c. but as n is exceeding great, $x = a + \frac{aL}{m}$,

and $x - a = \frac{aL}{m}$, and $Ar = \frac{H}{n}$. Let $n =$

1000 000, then will $rc = \frac{a}{m} \times \frac{B-A}{1000\ 000}$, and

$Ar = \frac{H}{1000\ 000}$; and the triangle Arc , may be

looked on as a right angled plain triangle. And

it will be, as $cr : Ar :: \text{rad. } (1) : \frac{Ar}{cr} = \tan. <$

Acr or $< B$ required $= \frac{mH}{a \times B - A}$.

Cor. When the angle at B is found, it will be, as
cof. B : BC :: rad. (1) : length of the spiral
BAQC, &c.

For $\frac{BC \times \text{rad.}}{\text{cof. B}} = \text{tangent at B, and (Prop. IV.$

Log. Spiral,) that tangent = whole length of the
 spiral.

PROB. XXXIV.

To find a point P within a parallelogram; through 114.
 which, if two lines AB, CD, be drawn parallel to
 the sides, the segments shall be in continual proportion,
 AP : CP : PB : PD.

Let

Fig.

114.

Let $AP = x$, $PC = y$, then $PB = \frac{yy}{x}$, $PD =$

$\frac{y^3}{xx}$, $AB = a$, $CD = b$. Then $x + \frac{yy}{x} = a$, or

$xx + yy = ax$. And $y + \frac{y^3}{xx} = b$, or $y \times \frac{xx + yy}{xx}$

$= bxx$. Hence $y \times ax = bxx$, and $ay = bx$, and

$y = \frac{bx}{a}$; therefore $ax = xx + yy = xx + \frac{bbxx}{aa}$,

and $a^2x = aaxx + bbxx$, or $aax + bbx = a^3$, and

$x = \frac{a^3}{aa+bb}$, and $y = \frac{aab}{aa+bb}$.

PROB. XXXV.

A ship sailed directly south, with a uniform motion, from lat. 37° , to lat. $37^\circ 48'$, in 22 hours; while the vane all the while was kept in the same position, N. E. by E. by a wind which moved 20 miles, or minutes an hour. I demand the true point of the wind.

115. Let $m = 48'$, $s = \frac{48}{22} = \frac{m}{22}$, the ship's motion

in an hour, $w = 20$, the wind's motion in an hour.

Let AB be the ship's way, AC the position of the vane, AD the direction of the wind. Draw CD parallel to AB , and DB parallel to AC . Because the vane is at rest as it moves through the wind, it acts upon neither side; and therefore the particles of wind that were in contact with the vane at A , in the position CA ; will also be in contact with the vane at D , in the position DB ; because these particles neither approach nor recede from the vane; otherwise they would move the vane one way or other.

The

ART. XVIII. PROBLEMS.

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The angle CAB is given, and therefore DBA is Fig. given, and in the triangle ADB, $AD(w) : AB(s) :: S.ABD (5 \text{ points or } 56^\circ 15') : S.ADB \text{ or } CAD = 5^\circ : 12'$. From CAB, $123^\circ 45'$, take $5^\circ 12' = CAD$, then $DAB = 118^\circ 33'$, the point of the wind.

Cor. Hence this rule, as the velocity of the wind AD : to the velocity of the ship AB :: so the sine of the angle between the ship's way and the position of the vane : to the sine of the difference between the true and apparent place of the wind.

PROB. XXXVI.

The circumference of the circle ABQ is 21; a string is put about it, and knotted at P, whose length is 25. To find the area described by drawing the knot P round about.

It is plain that P will describe a circle, whose center is C, the center of the first. Draw CD perp. to ACP, and CB perp. to BP the tangent at B. Then AD is a quadrant $= 5\frac{1}{4}$. Radius CD

or $CB = r = \frac{10.5}{3.1416}$. Let arch $DB = a$, then

$DBP = \text{sum of the arch and cotangent} = 7\frac{1}{4} = n$;

but the cotan. $BP = \frac{rr}{a} - \frac{a}{3} - \frac{a^3}{45rr} - \frac{2a^5}{945r^4}$

$- \frac{a^7}{4725r^6} \&c.$ Therefore $\frac{rr}{a} + \frac{2}{3}a - \frac{a^3}{45r^2} -$

$\frac{2a^5}{945r^4} - \frac{a^7}{4725r^6} = n$. And by reversion $a =$

1.8493 ; then $21 : 360 :: 1.849 : 31.7 \text{ degrees} = 31^\circ 42' = \angle DCB$, or BPC . Then $S.BPC (31^\circ 42') : CB (3.3422) :: \text{rad.} : CP = 6.551$, the radius of the circle described by P. Then $CP^2 \times 3.1416 = 134.82$ the area of that circle.

PROB.

117. If ADF be the arch of a circle, and supposing the arch AD continually evolves, beginning at A , the point A describing the curve ABG . To find the length AG , and area FAG .

Let the radius $CA = r$, arch $AD = z$. Take Dd infinitely small, and draw the tangents DB, db ; and make $AB = v$, $Bb = \dot{v}$, $Dd = \dot{z}$, then $DB = \text{arch } DA$, and the angle $BDb = DCd$, and the sectors DCd and BDb are similar; whence $CD (r)$:

$$DB (z) :: Dd (\dot{z}) : Bb (\dot{v}) = \frac{z\dot{z}}{r}, \text{ and } v \text{ or } AB =$$

$$\frac{zz}{2r}, \text{ and } AG = \frac{\overline{ADF}^2}{2r}.$$

Again, DB being perp. to AB , the fluxion of the area ABD is $= \frac{zv}{2} = \frac{z^2\dot{z}}{2}$; and the area $ABD = \frac{z^3}{6r}$, and area $AGF = \frac{AF^3}{6r}$.

P R O B. XXXVIII.

Suppose the value of a moor to be 4l. 8s. an aker; and walling will cost 6s. a rod, or 7 yards in length. To find the quantity of ground (in form of a square,) to be taken in, so that the walling may be just equal to its value.

Let x = side of the square in yards, $r = 1$ rod, or 7 yards long, p = price of walling a rod, a = an acre, m = value of it. Then $r : p :: \text{circumference } 4x : \frac{4px}{r} = \text{whole price of walling. And}$

$a :$

$a : m :: xx : \frac{mxx}{a}$ = whole value of the ground; Fig. 117.

then, per quest. $\frac{mxx}{a} = \frac{4px}{r}$, and $x = \frac{4pa}{rm} =$
 $\frac{24}{20 \times 7} \times \frac{4840}{4.4} = 188.6$ yards. And $\frac{188.6^2}{4840}$
 $= 7.35$ acres.

PROB. XXXIX.

To find a number (x), by which multiplying the difference of the bung and head diameters of a spheroidal cask, and then added to the lesser will be the mean diameter, or the diameter of a cylinder equal to the cask.

Let D = bung diameter, d = head diameter, then $D - d \times x + d$ = mean diameter. Then the solidity of the cask = $\frac{2D^2 + d^2}{3} \times .7854b =$
 $\overline{D - d \times x + d}^2 \times .7845x$, and $\overline{D - d \times x + d}^2 =$
 $\frac{2D^2 + d^2}{3}$, and $\overline{D - d \times x + d} = \sqrt{\frac{2}{3}DD + \frac{1}{3}dd}$,
 and $\overline{D - d \times x} = \sqrt{\frac{2}{3}DD + \frac{1}{3}dd} - d$, whence $x =$
 $\frac{\sqrt{\frac{2}{3}DD + \frac{1}{3}dd} - d}{D - d}$.

Cor. $\sqrt{\frac{2}{3}DD + \frac{1}{3}dd}$ is the mean diameter sought. And x is always between $\frac{2}{3}$ and $\sqrt{\frac{2}{3}}$, or between .666 and .816.

SCHOL.

After the same manner $\sqrt{\frac{DD + Dd + dd}{3}} =$
 mean diameter for the frustum of a cone. And x always between $\frac{1}{2}$ and $\sqrt{\frac{1}{2}}$, or between .5 and .578.

PROB.

118. To find the surface of the solid generated by a cycloid AMG revolving about its shorter axis AR.

Let $AR = a$, $AP = x$, arch $AB = s$, $PB = v = \sqrt{ax - xx}$, $PM = y$, $S =$ surface required, $AM = z$. By the nature of the cycloid, $zz = 4ax$, and $2z\dot{z} = 4a\dot{x}$, or $z\dot{z} = 2a\dot{x}$, whence $\dot{z} = \frac{2a\dot{x}}{z}$

$$= \frac{2a\dot{x}}{\sqrt{4ax}} = \dot{x} \sqrt{\frac{a}{x}}; \text{ and } y = v + s. \text{ Then}$$

$$\dot{S} = cy\dot{z} = \overline{v + s} \times c\dot{x} \sqrt{\frac{a}{x}} = c\dot{x} \sqrt{\frac{a}{x}} \times \sqrt{ax - xx} + sc\dot{x} \sqrt{\frac{a}{x}} = c\sqrt{a} \times \dot{x} \sqrt{a - x} + c\sqrt{a} \times sx^{-\frac{1}{2}}\dot{x}.$$

And the fluent $S = c\sqrt{a} \times \frac{-\frac{2}{3}}{a - x^{\frac{3}{2}}} \times \frac{a - x^{\frac{3}{2}}}{\frac{3}{2}} + \text{fl. } c\sqrt{a} \times sx^{-\frac{1}{2}}\dot{x} = -\frac{2}{3}c\sqrt{a} \times \frac{a - x^{\frac{3}{2}}}{\frac{3}{2}} + c\sqrt{a} \times 2sx^{\frac{1}{2}} + c\sqrt{a} \times 2a\sqrt{a - x}$; because $s = \frac{ax}{2v}$. And the fluent corrected, or $S = c\sqrt{a} \times$ into :

$$\frac{4a + 2x}{3} \sqrt{a - x} + 2s\sqrt{x}; \text{ and the whole surface} \\ = 2ca \times ABR - \frac{2}{3}a.$$

PROB. XLI.

119. Let ABE be a semicircle whose center is C; MD, DQ, two sides of a square. And let the square revolve about the circle, so that the side DM may always pass through the point E at the end of the diameter, whilst the other side DQ always touches the circle. To find the locus of the angular point D.

Upon the radius CE of the given circle, describe the semicircle CHE; and from the center C draw

draw CB to the point of contact B, which will be Fig. perp. to the tan. DB; and D being a right angle, 119. CB and ED are parallel. Through the center O, draw OHP parallel to ED or CB; then the angle $\text{EOH} = \text{ECB}$; and drawing HF perp. to EF; HF and BD will be tangents to the two circles, and parallel to one another. Therefore the figures EOHF, ECB D, are similar. And since $\text{EO} = \frac{1}{2} \text{EC}$, $\text{EF} = \frac{1}{2} \text{ED} = \text{FD}$, and $\text{EH} = \text{DH}$. Make $\text{HP} = \text{HO}$, and draw DP.

The triangles EHO, DHP, are similar and equal; for $\text{DH} = \text{EH}$, and $\text{PH} = \text{OH}$, and $\angle \text{DHP} = \angle \text{EHO}$, (they being the complements of the equal angles DHF, EHF), therefore $\text{PD} = \text{EO} = \text{PH}$. Whence if the circle DHNB be described with the radius PH or PD; the arches DH, EH, will be equal, because their cords are equal. And consequently if the arch DH revolves upon the equal arch EH, it will describe an epicycloid passing through D; or the point D in the square, will describe the same epicycloid, which will be the locus of all the points D.

PROB. XLII.

To find the length of the curve of the epicycloid VD. 120.

Let $\text{AB} = b$, $\text{AC} = n$, $\text{AV} = d$, $\text{VB} = 2r$, arch WD or GE, or $\text{BE} = s$. Sine $\text{Dm} = v$, vers. $\text{Wm} = x$, $\text{AD} = y$, $\text{Do} = \text{Ee} = \text{Ff} = s$, arch $\text{VD} = z$, then $d = b + 2r$, and $n = b + r$.

With the radius Ad , describe the arch *nod*. Then whilst the point of contact E is carried through the arch Ee , the line Ao is carried through the arch $\text{Ff} = \text{Ee}$, and therefore AD through the angle DAd . Draw CP perp. to AD. By the nature of the circle $vv = 2rx - xx$, and $yy = d - x^2 + vv =$

Fig. = $dd - 2nx$. And $vs = r\dot{x}$. The triangles ADm
 120. and ACP are similar, also CDP and Dno: there-
 fore AD (y) : Am ($d - x$) :: AC (n) : AP =
 $\frac{d-x}{y} n$. And DP = $y - \frac{d-x}{y} n = \frac{rd-nx}{y}$.

Again, CD (r) : DP ($\frac{rd-nx}{y}$) :: Do (s or $\frac{r\dot{x}}{v}$)
 : no = $\frac{rd-nx}{vy} \dot{x}$. Also AF (b) : Ao (y) :: Ff

(s or $\frac{r\dot{x}}{v}$) : od = $\frac{ry}{bv} \dot{x}$. Then $nd = \frac{ry}{bv} \dot{x} +$
 $\frac{rd-nx}{vy} \dot{x} = \frac{ryy + brd - bnx}{bvy} \dot{x} = \frac{2r-x}{bvy} dnx$.

Also $nD = y = \frac{-n\dot{x}}{y}$; whence Dd^2 or $\dot{z}^2 =$
 $\frac{2r-x^2}{b^2v^2y^2} d^2n^2\dot{x}^2 + \frac{n^2\dot{x}^2}{yy} = \frac{2r-x^2 \cdot dd - bbvv}{b^2v^2y^2} n^2\dot{x}^2$
 $= \frac{2r-x \cdot dd + bbx}{bbyyx} n^2\dot{x}^2 = \frac{2rdd - 4nrx}{bbyyx} n^2\dot{x}^2 =$

$\frac{dd - 2nx}{bbyyx} \times 2rn^2\dot{x}^2 = \frac{2rn^2\dot{x}^2}{bbx}$. Therefore $\dot{z} =$

$\frac{n\dot{x}}{b} \sqrt{\frac{2r}{x}}$, and $z = \frac{2n}{b} \sqrt{2rx} = \frac{2b + 2r}{b} \times$

cord WD.

PROB. XLIII.

121. To find the place of a planet in its orbit, or the an-
 gle at the other focus F, having the distance FN given.

Let S be the sun, AC = a , CE = c , CF = n ,
 SN = y , FN = v , M = mean motion, F = mo-
 tion round F, N the planet's place, $\dot{x}$ = perp.
 from n upon SN, and FN.

Then

Then (by Prop. II. Sect. V. Astronomy,) $\dot{M} = \frac{y\dot{x}}{c}$, and $\dot{F} = \frac{a\dot{x}}{v}$. And $M = \text{area } \frac{ASN}{c}$. To ^{Fig. 121.}

find the fluent of $\frac{a\dot{x}}{v}$; draw FD perp. to the tangent NT; then (by Cor. 2. Prop. XXXVI. Ellipsis,) $\sqrt{vy} : c :: v : FD = \frac{cv}{\sqrt{vy}}$; and the triangles

Non, NDF, are similar, and ND $\left(\sqrt{vv - \frac{ccv}{y}}\right)$

$$: DF \left(\frac{cv}{\sqrt{vy}}\right) :: \dot{v} : \dot{x} = \frac{c\dot{v}}{\sqrt{vy - cc}} =$$

$$\frac{c\dot{v}}{\sqrt{2a - v} \times v - cc}, \text{ and } \dot{F} \text{ or } \frac{a\dot{x}}{v} = \frac{a\dot{v}}{v\sqrt{2av - vv - cc}}$$

$$= \frac{acv^{-2}\dot{v}}{\sqrt{-1 + 2av^{-1} - ccv^{-2}}}, \text{ put } z \text{ for } v, \text{ to}$$

agree with the forms, then $\dot{F} =$

$$\frac{acz^{-2}\dot{z}}{\sqrt{-1 + 2az^{-1} - ccz^{-2}}}, \text{ which comes under the}$$

27th Form, where $p = \frac{aa}{cc} - 1 = \frac{nn}{cc}$, $v =$

$$a - \frac{cc}{z} = \frac{az - cc}{z}, \text{ and then } \frac{ac}{cc} \times p - \frac{vv}{cc} \Big|^{-\frac{1}{2}} \dot{v}$$

$$= \frac{av}{c} \times \frac{nn}{cc} - \frac{vv}{cc} \Big|^{-\frac{1}{2}} = \frac{a\dot{v}}{\sqrt{nn - vv}}. \text{ And this}$$

comes under the 10th Form. Put z again for v ;

$$\text{then } \dot{\phi} = \frac{a\dot{z}}{\sqrt{nn - zz}}; \text{ and by Form 10th, } \phi =$$

$a \times \frac{\text{the arch}}{n}$ whose sine is z , radius n . And re-

storing v , $\phi = a \times \frac{\text{arch}}{n}$ (sine $= v$). But because

Fig. 121. v was $= a - \frac{cc}{z}$, restore this, and $\phi = a \times \frac{\text{arch}}{n}$.

Sine $= a - \frac{cc}{z}$. And restoring the first v , $F =$

$\frac{a}{n} \times \text{arch}$, sine $= a - \frac{cc}{v}$, rad. $= n$. But in A,

where $F = 0$, $v = a - n$, and sine $= a - \frac{cc}{a-n}$

$= -f$, and the arch is a (negative) quadrant;

therefore by correction, $F = \frac{a}{n} \times \text{arch}$, whose

cosine is $\frac{cc}{v} - a$, and rad. $= n$.

P R O B. XLIV.

121. *Having given the distance FN of a planet from the upper focus F; to find the time of moving from the aphelion, from A to N, and the angle at F.*

Let $AC = a$, $CE = c$, $CF = n$, $SN = y$, $FN = v$. Then (by Prop. I. Sect. V. Astronomy,)

the area PSN $= \frac{.01745caD}{2} - \frac{c}{2} \sqrt{2ay - yy - cc}$,

where $D =$ degrees in the arch whose cosine is $\frac{a-y}{n}$, rad. $= 1$. But half the area of the ellip-

sis, $PEA = 180ca \times .01745$; therefore the area

$SAN = \frac{180ca \times .01745}{2} - \frac{.01745caD}{2} + \frac{c}{2}$

$\sqrt{2ay - yy - cc} = \frac{ca \times .01745}{2} \times 180 - D + \frac{c}{2}$

$\sqrt{2ay - yy - cc} = \frac{ca \times .01745}{2} d + \frac{c}{2} \sqrt{2ay - yy - cc}$,

where $d =$ degrees in the arch, whose cosine is $y - a$

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$\frac{y-a}{n}$ or $\frac{a-v}{n}$. But area AEPB (180ca x .0174) ^{Fig. 121.}

: periodical time (t) :: area SAN $\left(\frac{ca \times .0174}{2} d + \frac{c}{2} \sqrt{2ay - yy - cc} \right)$: time of describing SAN.

Or $360 : t :: d + \frac{\sqrt{2ay - yy - cc}}{a \times .0174} :$

$d + \frac{\sqrt{2ay - yy - cc}}{.017453a} \times \frac{t}{360}$, or $d + \frac{\sqrt{2av - vv - cc}}{.017453a} \times$

$\frac{t}{360}$ = time of describing SAN.

Again, If $z = \cos. PSN$, then (Astron. *ib.*) y or

$2a - v = \frac{cc}{4 + zn}$, from whence $z = \frac{cc - 2a - v \times a}{2a - v \times n}$.

And (by Trigonometry) $NF (v) : S. FSN (\sqrt{1 - zz}) :: NS (2a - v) : S. \text{angle SFN or AFN}$

$= \frac{2a - v}{v} \sqrt{1 - zz}$.

Hence if v be given, the time of describing AN from the aphelion will be found.

And from v being given, z is found; and then the angle at the upper focus F.

Cor. If you say as t the periodic time : 360 :: so the time in SAN, before found : to the angle AFN; then it will appear how much this value of AFN, differs from the true value before found.

PROB. XLV.

Given the two sides of the triangle ACB, $AC = 16$, ^{122.} $AB = 25$. To find CB so that a body may descend through it, in the least time possible.

I i 2

Let

Fig. Let $AB = b$, $AC = d$, AB being parallel to the horizon, draw COD perp. to AB , $x = S.CAB$, $y =$ its cosine. Then rad. $(1) : AC (d) :: S.ACB (y) : AO = dy$. And rad. $(1) : AC (d) :: S.CAB (x) : CO = dx$. Then $CB = \sqrt{bb + dd - 2bdy}$, let BD be perp. to BC , then $CD = \frac{CB^2}{CO} = \frac{bb + dd - 2bdy}{dx}$. Now a body will descend through CD in the same time it descends through CB ; therefore CD is a minimum, or $\frac{bb + dd - 2bdy}{dx} = m$, in Fluxions, $-dx \times 2bdy - dx^2 \times \frac{2bdx}{bb + dd - 2bdy} = 0$, or $-2bdxy = \frac{bb + dd - 2bdy}{y} \times x$; but $xx + yy = 1$, and $xx + yy = 0$, and $y = \frac{-xx}{y}$, therefore $\frac{2bdx^2}{y} = \frac{bb + dd - 2bdy}{y} \times x$, and $2bdxx = \frac{bb + dd - 2bdy}{y} \times y$, or $2bdxx = \frac{bb + dd}{y} \sqrt{1 - xx} - 2bd \times 1 - xx = \frac{bb + dd}{y} - 2bd + 2bdxx$, and $\frac{bb + dd}{y} = 2bd$, and $y = \frac{2bd}{bb + dd} = \cos. 24^\circ 46'$. And the angle C will be obtuse.

P R O B. XLVI.

123. *There is given the base AB of the isosceles triangle ADB , and the part DG as far as the perp. AG . To find the sides AD , DB .*

Let AD or $DB = x$, AE or $EB = b$, DE being perp. to AB , and $GD = a$. Then $AB^2 = AD^2 + DB^2 + 2GDB$, or $4bb = 2xx + 2ax$, and $xx + ax = 2bb$. And $x + \frac{1}{2}a = \sqrt{2bb + \frac{1}{4}aa}$, or $x = \sqrt{2bb + \frac{1}{4}aa} - \frac{1}{2}a$.

P R O B.

PROB. XLVII.

To find the fluent of $\frac{\dot{y}}{y} - \frac{\dot{x}}{x} = \frac{x^m \dot{x}}{ay^n}$.

Multiply by y^n , then $y^{n-1} \dot{y} - \frac{y^n \dot{x}}{x} = \frac{x^m \dot{x}}{a}$.

Assume the fluent $\frac{y^n}{x^r} = \frac{sx^p}{ap}$; this in Fluxions is

$$\frac{ny^{n-1} \dot{y}}{x^r} - \frac{ry^n \dot{x}}{x^{r+1}} = \frac{sx^{p-1} \dot{x}}{a}, \text{ multiply this by } \frac{x^r}{n}$$

$$\text{then } y^{n-1} \dot{y} - \frac{ry^n \dot{x}}{nx} = \frac{sx^{p+r-1} \dot{x}}{na};$$

$$\text{compare this with } y^{n-1} \dot{y} - \frac{y^n \dot{x}}{x} = \frac{x^m \dot{x}}{a},$$

to make them the same, then $\frac{r}{n} = 1$ or $r = n$,

$\frac{s}{n} = 1$ or $s = n$; lastly, $p + r - 1 = m$, and

$p = m - r + 1 = m - n + 1$. Whence the fluent is $\frac{y^n}{x^n} = \frac{nx^{m-n+1}}{ap}$, or $\frac{ay^n}{x^n} = \frac{nx^{m-n+1}}{m-n+1}$.

And when x and $y = a$, then $a = \frac{nam-n+1}{m-n+1}$, or

$$a^{m-n} = \frac{m-n+1}{n}.$$

PROB. XLVIII.

If the curve RDB be of such a nature, that the tangent DT revolving uniformly about the curve, the particle of the curve, at any place D, may be as the square of the sine of the angle ADt. To find the nature of the curve.

Let $RA = x$, $AD = y$, $RD = z$. Then since DT is supposed to change its direction uniformly,

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there-

Fig. therefore the angle of contact is given, and therefore z' is as the radius of curvature. But $S. <$

$$ADt = \frac{\dot{x}}{z}; \text{ therefore } z' \propto \frac{\dot{x}\ddot{x}}{z\dot{z}}; \text{ whence } \frac{\dot{x}\ddot{x}}{z\dot{z}}$$

$\propto$ rad. curvature $\frac{\dot{z}\ddot{y}}{\dot{x}}$ ($\dot{z}$ being given). Whence

(assuming the given quantity a) ($\dot{x}\ddot{x} = \frac{\dot{z}\ddot{y}}{a}$, and

the fluent is $\frac{\dot{x}^3}{3} = \frac{\dot{z}\ddot{y}}{a}$, which is correct, for in R,

$y = 0$, and $\dot{x} = 0$, whence $\dot{x} \sqrt{\frac{a}{3}} = \dot{z}\sqrt{y}$. For

brevity's sake put $\frac{a}{3} = 1$, then $\dot{x} = y^{\frac{1}{2}}\dot{z}$. But

$\dot{x}^2 = y^{\frac{1}{2}}\dot{z}^2 = y^{\frac{1}{2}}\dot{x}^2 + y^{\frac{1}{2}}\dot{y}^2$, and $1 - y^{\frac{1}{2}} \times \dot{x}^2 = y^{\frac{1}{2}}\dot{y}^2$,
therefore $\dot{x} = \frac{y^{\frac{1}{2}}\dot{y}}{\sqrt{1 - y^{\frac{1}{2}}}}$. Put $v = y^{\frac{1}{2}}$, and $v^2 = y$,

then $\dot{x} = \frac{2v^2\dot{v}}{\sqrt{1 - vv}}$; and the fluent is $x = 2 -$

$\frac{2 + vv}{1} \sqrt{1 - vv}$, for the nature of the curve RDB.

Reduced $4x - xx = 3v^4 + v^6 = 3y^{\frac{3}{2}} + 4y^{\frac{3}{2}}$.

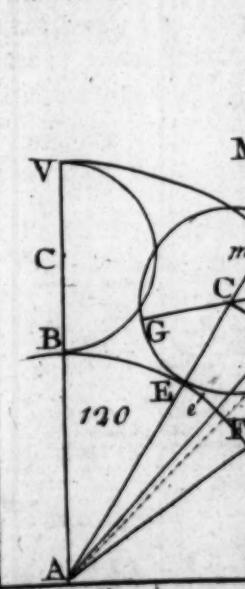
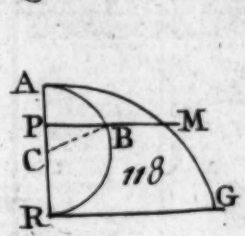
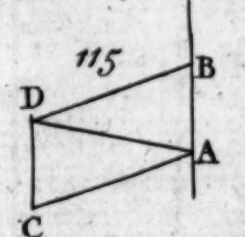
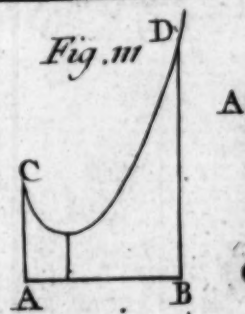
Cor. 1. If C be the center of the curve, then CR = 2CB. For when y is greatest, $\dot{x} = \dot{z}$, and therefore $\dot{x} (\dot{z}) = y^{\frac{1}{2}}\dot{z}$, and $y^{\frac{1}{2}} = 1$, or $y = 1$. Whence $v = 1$, and $\sqrt{1 - vv} = 0$, therefore $x = 2$; that is, CB = 1, CR = 2.

Cor. 2. CB = $\frac{1}{3}a$.

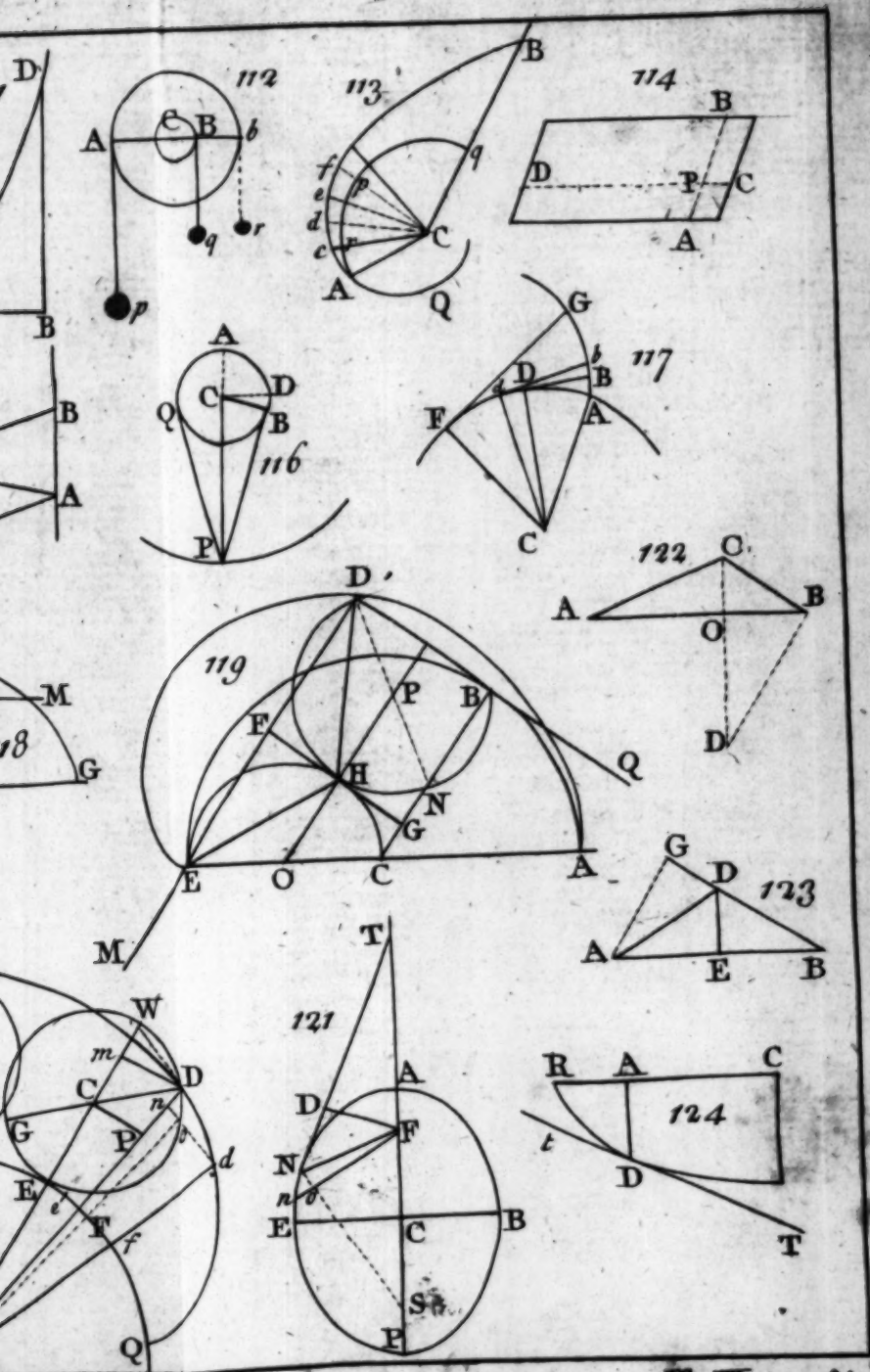
Cor. 3. To construct the curve, take any angle, find its sine $\frac{\dot{x}}{z}$, or $y^{\frac{1}{2}} = v$. Then we have $y = v^2$, and

$x = 2 - \frac{2 + vv}{1} \sqrt{1 - vv}$; and finding these for

any



Miscellaneous



any series of angles, a table may be made of x and y Fig. from these angles, and the curve constructed.

Otherwise assuming any value for y finds v , and v finds x .

PROB. XLIX.

Suppose AC, Ac , to be curves of the same kind, ^{125.} whose vertices are in A , and their axes AF . To find the nature of the curve FCE , that shall cut all these curves AC, Ac , &c. at right angles.

Let two curves AC, Ac , be infinitely near, Cc a particle of the curve FC , between them. Draw the ordinates BC, bc ; and also DC, dc .

Let AB or $DC = x$, BC or $AD = y$, $ED = v$, $AE = d$, $oC = \dot{v}$, $oc = \dot{x}$. Draw the tangent CT to the curve CF , and the tangent CQ to the curve Ac . Then the triangles Coc , CBQ are similar, and subtangent $QB \left(\frac{y\dot{x}}{\dot{y}} \right) : BC \text{ (} y \text{ or } d - v \text{)} :: Co (\dot{v}) : oc (\dot{x})$, whence $\frac{y\dot{x}\dot{x}}{\dot{y}} = \overline{d - v} \times \dot{v}$, and $d\dot{v} - \frac{1}{2}v\dot{v} = \text{fl. of } \frac{y\dot{x}^2}{\dot{y}}$.

EXAM. I.

Let AC, Ac , be parabola's. Then $BQ = \frac{y\dot{x}}{\dot{y}} = 2x$, and $d\dot{v} - \frac{1}{2}v\dot{v} = \text{fluent of } 2x\dot{x} = xx$, or $xx = \frac{1}{2} \times 2d\dot{v} - v\dot{v}$, an equation to an ellipsis, therefore FCE is an ellipsis. And if $AE = 10$, then $AF = \frac{10}{\sqrt{2}} = 7$ nearly.

EXAM. 2.

Let AC, Ac , be ellipses, whose transverse is given $= 2a$, then $2ax - xx = byy$, and $a\dot{x} - x\dot{x} =$
I i 4 =

Fig. 125. = byj , whence $\frac{y\ddot{x}\dot{x}}{j} = \frac{y\ddot{x}\dot{x} \times by}{a\dot{x} - x\dot{x}} = \frac{byy\dot{x}}{a-x} =$

$$\frac{2ax - xx}{a-x} \dot{x} = \frac{aa}{a-x} \dot{x} - a\dot{x} + x\dot{x}, \text{ and the flu-}$$

ent is $dv - \frac{1}{2}vv = 2.3025aa \times \log. \frac{a}{a-x} - ax + \frac{1}{2}xx$, for the equation of the curve.

In the hyperbola $\frac{y\ddot{x}\dot{x}}{j} = \frac{2ax + xx}{a+x} \dot{x} = a\dot{x} + x\dot{x} - \frac{aa\dot{x}}{a+x}$, and the fluent, $dv - \frac{vv}{2} = ax + \frac{1}{2}xx - 2.3025aa \times \log. \frac{a+x}{a}$.

PROB. L.

To find the sum of the series $z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \&c.$

Here some value must be put for z , as suppose

1. Then the series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \&c.$

that is, (collecting two terms into one) $\frac{2}{1.3} +$

$\frac{2}{5.7} + \frac{2}{9.11} + \&c.$ or $2 \times$ into $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11}$

$\&c.$ But by Prob. XXVIII. Increments, the sum of the series $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \&c. = T + \frac{1}{4v}$

$+ \frac{4}{4v} A + \frac{2.3.4}{6.v} B + \&c. = T + \frac{1}{4.41} +$

$\frac{4}{4.45} A + \frac{2.3.4}{6.49} B + \frac{3.5.4}{8 \times 53} C + \&c.$ where $T =$

the

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the sum of 10 terms = .38644, $v = 41$, $v = 45$, Fig. 125.
 $v = 49$, &c. and the whole sum = .39270, and
 $2 \times .39270 = .7854$, the sum of the given series.

PROB. LI.

To find the sum of the series $2\sqrt{d} \times$ into $1 +$
 $\frac{1.1}{2.5} A + \frac{3.5}{4.9} B + \frac{5.9}{6.13} C + \frac{7.13}{8.17} D$, &c. being
the fluent of $dd - xx^{-\frac{1}{2}} \times dx^{-\frac{1}{2}} \dot{x}$, when $x = d$; which is
the time of descent of a body in a quadrant, whose ra-
dius is d .

Here the z^{th} term is $\frac{2z-3}{2z-2} \times \frac{4z-7}{4z-3} s =$
 $\frac{z-\frac{3}{2}}{z-1} \times \frac{z-\frac{7}{4}}{z-\frac{3}{4}} s = s$; put $y = z - 1$, or $z =$
 $y + 1$, then $z = y = 1$. And the equation is now
 $\frac{y-\frac{1}{2}}{y} \times \frac{y-\frac{3}{4}}{y+\frac{1}{4}} s = \frac{y-\frac{1}{2}}{y} \times \frac{y-\frac{3}{4}}{y-\frac{3}{4}} s$, s being
the $z + 1^{\text{th}}$ term; according to the directions in
Prop. XV. Increments.

To find the integral, compare this with Ex. 26.
Prop. XIII. Increments, and we have $m = \frac{1}{2}$, $n = \frac{3}{4}$,
 $z = y = 1$. And we shall have the integral

$z - n$, $s \times$ into $-\frac{1}{m} + \frac{\frac{n}{z} P}{m+1} + \&c.$ Now

this series being negative, does not give the sum of
the series to the z^{th} term, but the sum of all the
terms, *ad infinitum*, beyond the z^{th} term. For the
greater z is, the less the sum will be. Therefore
sum up 4 terms of the series = 1.16570, and the
5th term will be .01608; and writing y for z , we
shall have for 4 terms 1.16570, and the rest *ad in-*
finitum

Fig.

$$125. \text{ finitum} = \overline{y-n} \times s \times \frac{1}{m} + \frac{\frac{n}{y} P}{m+1} + \frac{\frac{n+1}{y+1} Q}{m+2}$$

$$+ \frac{\frac{n+2}{y+2} R}{m+3} \text{ \&c. that is (putting } y=5) 4\frac{1}{4} \times .01608$$

$$\times \frac{1}{0\frac{1}{2}} + \frac{\frac{3}{4.5} P}{1\frac{1}{2}} + \frac{\frac{7}{4.6} Q}{2\frac{1}{2}} + \frac{\frac{11}{4.7} R}{3\frac{1}{2}} + \frac{\frac{15}{4.8} S}{4\frac{1}{2}} \text{ \&c.}$$

P =	.06834	2P =	.13668
Q =	.01025	$\frac{2}{3}Q =$	683
R =	299	$\frac{2}{3}R =$	119
S =	117	$\frac{2}{7}S =$	33
T =	55	$\frac{2}{9}T =$	12
U =	29	$\frac{2}{11}U =$	5
X =	17	$\frac{2}{13}X =$	3
Y =	10	$\frac{2}{15}Y =$	2
		sum =	.14525

series .14525
add the first 4 terms 1.16570

total sum 1.31095
multiply by $2\sqrt{d}$

2.62190 $\sqrt{d}$ for the sum of all.

Otherwise.

It may likewise be solved by *Sterling's Series*, Prop. VIII. Here the equation of the series is

$$T' = \frac{z - \frac{1}{2}}{z} \times \frac{z - \frac{3}{4}}{z + \frac{1}{4}} T, \text{ and the successive va-}$$

lues of z are 1, 2, 3, 4, &c. and $m = \frac{1}{2}$, $n = \frac{3}{4}$, sum up the 4 first terms of the series = 1.16570, and the fifth term or $T = .01609$, and $z = 5$.

$$\text{Then } A = 4\frac{1}{4}T, B = \frac{3}{4.5} A, C = \frac{7}{4.6} B, D =$$

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$\frac{11}{4.7} C, E = \frac{15}{4.8} D, \&c.$ and $S = 2A + \frac{2}{3}B + \frac{2}{3}C$ Fig.
 $+ \frac{2}{3}D + \frac{2}{3}E, \&c.$ which brings out the same answer as before.

PROB. LII.

Suppose the bob of a pendulum ABC to be in the 126. form of a paraboloid, whose axis is BD. To find the center of oscillation.

Let $BD = x, AD = y, 2rx = yy.$ And suppose BD parallel to the axis of motion XY. Also let $c = 3.1416, Df = z.$ Then (by Prob. XVIII. Sect. II. Fluxions,) in the circle DA the fluxion of the figure is $2c \times Df \times Df^2 = 2cz^3\dot{z},$ and the fluent $= \frac{1}{2}cz^4 = \frac{1}{2}cy^4,$ at last, $= \frac{1}{2}c \times 4rrxx,$ and $\dot{G} = 2crrxx\dot{x},$ and $G = \frac{2}{3}crrxx^3.$ But the solidity of the

paraboloid $ABC = \frac{cyyx}{2} = B = crxx;$ therefore

$$\frac{G}{dB} = \frac{2crrxx^3}{3crxxd} = \frac{2}{3} \times \frac{rx}{d} = \frac{yy}{3d}.$$
 And the distance

from the axis of motion $= d + \frac{1}{3} \times \frac{yy}{d}.$

PROB. LIII.

To find the density of the sun, compared with that of the earth.

Let $D =$ density of the sun, $d =$ density of the earth, $P =$ earth's parallax as seen from the sun $=$ sun's apparent diameter seen from the earth, $p =$ moon's mean horizontal parallax, $T =$ a year, or the earth's periodical time, $t =$ moon's periodical time. Then (by Prop. XIX. Centripetal Forces) the densities of central bodies are reciprocally as the cubes of the parallaxes, and squares of the

Fig. 126. the periodical times; that is, $D : d :: \frac{1}{P^3 T^2} :$

$$\frac{1}{p^3 t^2} :: p^3 t t : P^3 T^2, \text{ and } D = \frac{p^3 t t d}{P^3 T^2}.$$

Cor. Hence the density of the sun is bad, without either the sun's distance or his parallax.

P R O B. LIV.

The latitude of the place being given; to find the time of shortest twilight.

It is found by observation, that the time of day-break is when the sun is 18 degrees below the horizon.

127. Let ZHNO be the meridian, HO the horizon, EQ the equinoctial, BD the parallel of day-break, Pp the earth's axis, Z the zenith, N the nadir, *os* the sun's parallel of declination when twilight is shortest. Draw the parallel *rv* infinitely near *os*, and draw the hour circles *op*, *rap*, *tsp*, *vp*. Now when the portion of the parallel *os*, comprehended between HO and BD is the shortest, it will be equal to the portion *rv*; because when it is the least, it neither increases nor decreases; therefore *oa* the decrease at *o*, is equal to *tv* the increase at *s*. But in the triangles *oar*, *tsv*, *ar = st*, and *oa = tv*, and the angles at *a* and *t* are right, therefore the angle *roa = angle tsv*. But by Trigonometry the angle *poN* of the spherical triangle *poN* is $= roa$. And the angle *pvN*, of the spherical triangle *pvN* is $= tsv$. Therefore to find the angles of these spherical triangles. Let $\text{rad.} = 1$, $x = s$. declination $= \cos. po$, or *pv*, $y = s$. *po*, or *pv*, $l = s$. latitude $= \cos. pN$, $s = s$. $18^\circ = \cos. Nv$, $c = s$. *Nv*. Then (by Case 11th, Spherical Trigonometry) $\cos.$

$$poN \text{ (or } roa) = \frac{l}{1 \times y}, \text{ because the cosine of No}$$

(being

(being a quadrant) is o . Also the cof. pvN (or tvS) Fig. $= \frac{l-sx}{cy}$; therefore $\frac{l}{y} = \frac{l-sx}{cy}$, and $cl = l-sx$, ^{127.}

therefore $sx = l-cl$, and $x = \frac{1-c}{s} l$; but $1-c$ is the versed sine of 18° . Therefore as $s. 18^\circ$: vers. 18° :: $s. \text{latitude}$: $s. \text{declination of the sun south, when twilight is the shortest.}$

Cor. Hence, as radius : tan. 9° :: $s. \text{latitude}$: $s. \text{sun's declination south, when twilight is shortest.}$

For (by Cor. I, Prop. II. Trigonometry) radius : tangent of an arch :: sine of 2 arch : vers. twice the arch; that is, rad. : tan. 9° :: $s. 18^\circ$: vers. 18° :: $s. \text{latitude}$: $s. \text{declination.}$

It is said J. Bernoulli was 5 years in solving this Problem, and did it but very audwardly at last.

PROB: LV.

To find the proportional heat of the sun in all latitudes, for any day.

The action of the sun's rays is, as all other impulses or strokes, more or less forcible, according to the sines of the angles of incidence; whence the vertical ray (giving the greatest heat) being put equal to radius, the force of any other ray, upon the horizon, will always be as the sine of the sun's altitude. This being granted, it will follow, that the time of continuance of the sun's shining being taken for a *basis*, and the sines of the sun's altitudes erected thereon as perpendiculars, and a curve drawn through their extremities; the area of that curve will be proportional to the heat of the sun in that time.

Let AB be the length of a winter day, EQ the ^{128.} length of an equinoctial day, CD the length of a summer day. Or we may suppose CD the diurnal arch,

Fig. arch, reckoned on the equinoctial, to represent the 128. time.

Let l, m = sine and cosine of latitude; d, c = the sine and cosine of the sun's declination; z = PF the arch from noon, y, x = sine and cosine of PF; rad. = 1. Then by Astronomy, rad. : tan. latitude :: tan. declination : s. ascensional difference EA (or EC); then AF the semidiurnal arch is known = EF — EA, (or EF + EC) let n = s. AF, or CF.

Then GF is the sine of the meridian altitude. And to find MP the sine of the altitude, at the distance of time FP from noon, we have two sides and the included angle of a spherical triangle, to find the third side, therefore (by Case 8th, Spherical Trigonometry) $mcx \pm ld$ = cos. zenith distance = s. altitude MP. Then $MP \times \dot{z}$ or $mcx\dot{z} \pm ld\dot{z}$ = flux. of the area GFPM. But by the nature of the circle $x\dot{z} = r\dot{y} = \dot{y}$; whence the flux. area = $mc\dot{y} \pm ld\dot{z}$, and $mc\dot{y} \pm ld\dot{z}$ = area FGMP. And $mcn \pm ld \times AF$ = area AGF; and $mc \times n - y \pm ld \times AP$ = area AMP, the heat generated in the time AP, using + for summer, and — for winter.

Hence the heat generated in the whole day is = $2mcn \pm ld \times CD$, for summer; and $2mcn - ld \times AB$ for winter.

Cor. 1. Under the equinoctial the beat is $2crr$; and when the sun has no declination, the beat is $2r^3$.

Cor. 2. Under the pole the beat of a summer day is $rd \times 2EQ$.

Cor. 3. Under the frigid zones, where the sun sets not, the beat is $ld \times 2EQ$; and therefore is $2EQ \times$ sine of the sun's altitude at 6.

Cor. 4. In any latitude when the sun is in the equinoctial, the beat is $2mrr$.

Cor. 5. The greatest beat under the equinoctial, to the greatest beat under the poles; is as $2r^3$ to $rd \times 2AE$,

2AE, or as r to $d \times 3.1416$; or as r to $.3987 \times$ Fig.
 3.1416 ; that is, as 4 to 5. 128.

Cor. 6. In the latitude of London, the heat of the longest summer's day, to that of the shortest winter day, is as $6\frac{1}{2}$ to 1.

S C H O L.

Dr. Halley, in solving this Problem, remarks, that since the nature of heat is to remain in the subject, after the cause that heated it is removed; and particularly in the air under the equinoctial, the 12 hours absence of the sun, does very little still the motion impressed, by the past action of the rays wherein heat consists, before he arise again. But under the pole, the long absence of the sun for 6 months, wherein the extremity of cold does obtain, has so chilled the air, that it is as it were frozen, and cannot, before the sun has got far towards it, be any way sensible of his presence; his beams being obstructed by the thick clouds, and perpetual fogs and mists, and by that *atmosphere of cold*, as the late honourable Mr. Boyle was pleased to term it, proceeding from the everlasting ice, which in immense quantities does chill the neighbouring air, and which the too soon retreat of the sun leaves unthawed, to encrease again during the long winter that follows this short interval of summer.

But the different degrees of heat and cold in different places, depend in a great measure upon the accidents of the neighbourhood of high mountains, whose height exceedingly chills the air brought by the winds over them; and of the nature of the soil, which variously retains the heat; particularly the sandy, which in *Africa, Arabia*, and generally where such sandy deserts are found, do make the heat of the summer incredible to those that have not felt it.

To which I may add, that the sine of the angle of incidence, or obliquity of the rays, is not the sole

Fig. sole cause of the proportion of heat. But the
 128. quantity of atmosphere through which the light
 passes, has got a great effect upon it; for the more
 atmosphere it comes through the more light will be
 destroyed, and consequently the heat diminished
 upon that account, though the angle of incidence
 was to remain the same. Therefore it will follow
 that the quantity of heat will be less, when the
 sun is lower than in the ratio of the sine of in-
 cidence, as supposed in this calculation. See
 Prop. XVIII. B. I. Optics.

P R O B. LVI.

*To find the proportional heat of the sun for a year
 in any latitude, as London.*

Take a summer and a winter day where the sun's
 declination is the same; then by the last Prob. the
 heat of the

$$128. \quad \begin{aligned} \text{summer day} &= 2mcn + ld \times CD \\ \text{winter day} &= 2mcn - ld \times AB \\ \text{sum} &= \frac{4mcn + ld \times 4AE}{4}, \text{ or taking a} \end{aligned}$$

quarter thereof, the heat for these two days is =
 $mc \times \cos. AE + ld \times AE$, where AE is the ascen-
 sional difference. And if this be done every day
 from the equinox to the solstice, the sum of all is
 the heat, for half a year.

Now to approximate to the sum of these, find
 the ascensional difference for every 10 or 15 degrees
 of the ecliptic thus; let $t = \tan. \text{latitude}$, $s =$
 $s. \text{sun's longitude}$, $p = s. 23\frac{1}{2}$ the obliquity of the
 ecliptic; then $\text{rad. } (1) : s :: p : ps = d$ the $s. \text{decli-}$
 nation , and $\text{rad. } (1) : t :: \tan. \text{decl. } \left(\frac{ps}{c}\right) : \frac{pts}{c} :$
 $s. \text{ascen. difference}$, which gives AE and its cosine.
 Thus for 0, 15, 30, 45, &c. degrees of the eclip-
 tic, the declination, and ascensl. difference, and
 the

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the other requisites will be as in the following Fig. 128.

☉ lon- gitude.	decli- nation.	ascen. differ.	cof. decl. × cof. asc.	S. decl. × asc.
0	0	0	1.0000	0
15	5 55	7.483	.9862	.7723
30	11 30	14.816	.9473	2.9537
45	16 23	21.68	.8915	6.1128
60	20 12	27.55	.8321	9.5138
75	22 39	31.63	.7857	12.1827
90	23 30	33.13	.7505	13.2107

Then by the rule for 7 ordinates (Cor. 1. Prop. IX. Differential Method) we have,

$$\left. \begin{array}{l} A = 1.7505 \\ B = 1.7719 \\ C = 1.7794 \\ D = .8915 \end{array} \right\} \times m, \quad \left. \begin{array}{l} A = 13.2107 \\ B = 12.9550 \\ C = 12.4675 \\ D = 6.1128 \end{array} \right\} \times l \times .01745$$

And the sum of the first series = 50.14 } where L =
And sum of the second series = 7.99 } 91 days.

total 58.13, and this × 8,
for the reasons before given, gives 465.04 the heat
in a whole year in lat. $51^{\circ}\frac{1}{2}$.

Cor. 1. Under the equinoctial, $l = 0$, $m = r$, and
cof. AE = r . Therefore only the sum of all the crr
must be calculated.

☉ lon.	☉ decl.	cof. decl.
0	0°	1.0000
15	5 55	.9946
30	11 30	.9799
45	16 23	.9594
60	20 12	.9385
75	22 39	.9229
90	23 30	.9170

K k

Then

Fig.
128.

$$\begin{array}{l} \text{Then } A = 1.9170 \\ B = 1.9175 \\ C = 1.9184 \\ D = .9594 \end{array} \left. \vphantom{\begin{array}{l} A \\ B \\ C \\ D \end{array}} \right\} \times rr \text{ or } 1.$$

Then by the foresaid rule, the sum of all is 88.48, which $\times$ by 8, gives 697.8 for the equinoctial heat in a year. And hence the heat under the equinoctial, to that at London, is as 3 to 2.

Cor. 2. Under the poles, $m = 0$, and $l = r$, and AE must be supposed a quadrant $= q = \frac{3.1416}{2}$. Therefore we must find the sum of all the rqd .

☉ lon.	☉ decl.	S. declin.
0	0	0
15	5 55	.1031
30	11 30	.1993
45	16 23	.2821
60	20 12	.3453
75	22 39	.3851
90	23 30	.3987

$$\begin{array}{l} \text{Then } A = .3987 \\ B = .4882 \\ C = .5446 \\ D = .2821 \end{array} \left. \vphantom{\begin{array}{l} A \\ B \\ C \\ D \end{array}} \right\} \times rq \times 8.$$

And by the said Rule the sum of all $= 67.375 \times 4 = 269.5$, for the polar heat in a year; or if you will in half a year; for in the other half year, it gets none. And therefore the heat at London, to that under the Pole, is nearly as 12 to 7.

Cor. 3. And hence the equinoctial heat is to the polar heat, as 13 to 5 nearly.

P R O B. LVII.

In the stereographic projection of the sphere, to draw a lesser circle perpendicular to the plane of projection, which lies very near its parallel great circle.

When

When the scheme is large, as in the drawing of Fig. maps, and the circle lies very near its primitive great circle, it is not practicable to draw such a circle with a pair of compasses, as directed by the rules of that projection. And therefore it must be done after some other manner, as follows.

1 Way.

Let ADBE be the primitive, DCE the parallel great circle, GHI the circle to be drawn. Draw GN, *In* perp. to DE. It is plain by the nature of this projection, that NG or *nl* is the sine of DG, the circle's distance from its parallel great circle DE; also CH is the tangent of half DG; and these being set off from DE will give the three points G, H, I, through which describe the portion of the circle GHI with a bow, as directed in the Scholium to Prop. XII. of the Projection.

2 Way.

But when more exactness is required, several points must be found, through which to draw the circle.

Let s = nat. sine, t = tangent, f = secant of the arch GA, rad. = 1. Then rad. FH = t , FC = f , by Prop. V. Projection. And in the triangle FGC, FG (t) : S.GA or $\angle C$ (s) :: GC (1) : S.GFC or arch HG in degrees. Then let z = sine, v = versed sine, of any arch Hg less than HG; then tz = sine, and vz = versed sine of Hg to the radius t , let m = tan. of half DG. Then make Cl = tz , and perp. thereto make $lg = m + tv$, and g will be in the curve. And thus find as many points g as you will, and draw a curve line thro' them.

3 Way.

Draw the tangent HO, and put Hp = n , pg = x , and diameter 2FH = $2t$, then by the property of the circle Hp² = pg × 2FH — pg, that is, $nn = x \times 2t - x$, or $2tx - xx = nn$, and $tt - 2tx + xx =$

K k 2

 $tt - nn;$

Fig.

$$129. \quad tt - nn; \text{ whence } x = t - \sqrt{tt - nn} = \frac{nn}{2t} + \frac{n^4}{8t^3}$$

$= pg$, and g is in the curve; in most cases $\frac{n^4}{8t^3}$ may be left out. And thus any number of points may be found, thro' which the curve must be drawn.

4 Way.

130. This may be performed mechanically thus. Having found as before, the 3 points G, H, I , take a thin plane, as a strong paper or pasteboard, PQR , whose edge PR is very streight, lay that edge to the three points IHD , and (supposing the lines DH, HG , drawn) make the angle $PST = DHG$, and cut off the part PST . Then apply the side ST (of the instrument $TSRQ$) to the point G , and the side SR to the point I , and where the angle S falls, make a point g . Then shifting the instrument, still keeping the points G, I , in the sides ST, SR ; the angle S will find another point g ; and by this means you may find as many points as you will, through which to draw the curve.

Note, the curve may be drawn by a continued motion, thus. Stick up two pins at G and I , and move the instrument about, so that the side ST may continually slide by the pin G , and the side SR by the pin I ; then the point S will describe the curve required.

By this method there will be required a new instrument to be made for every parallel. And this method will describe any parallel, whether near to the parallel great circle, or far off.

P R O B. LVIII.

131. *Having given the curve ZFO, by the revolution of which about its axis ZH, a dome or cupolo is formed. To find the exterior curve SGg, so that all the arch stones*

stones lying aslope, the joints may be perpendicular to the inner curve ZF; and the whole may stand in equilibrio. 131.

Let HZFO, HZLo, be two plains intercepting a portion of the cupolo, which is constructed upon this supposition, that the arch stones, as IF, run quite through the thickness of the arch; in that position, and may freely slide up or down upon the joint.

1. When the joints meet in one point H, the curve ZF being a circle. Suppose the plane NDB drawn parallel to the horizon. The weight of the arch stone FGtf, by its weight tends in a line parallel to ZH; and the pressures against the joints FH, fH, are perpendicular to these joints. Therefore these three forces, the weight and two pressures, will be as three lines drawn perpendicular to their directions; that is, as Dd, DH, dH. And this would be so every where, if the depth FL was every where the same, but it grows less towards Z; in proportion to FM, or DB, or nd, which is the depth of the pyramid HFl at d. Therefore if the curve SGg be such, that the solid or arch stone FGrtf be always equal to the correspondent pyramid HDn; then G will be in the curve required. But the three pyramids HDn, HFl, and HGt, are as HD³, HF³, and HG³; whence HD³ : HG³ — HF³ :: pyramid HDn : solid FGrtf. Therefore if HD³ = HG³ — HF³ (or HG = $\sqrt[3]{HD^3 + HF^3}$), then the pyramid HDn = arch stone Ft. Consequently taking FG = $\sqrt[3]{HD^3 + HF^3} - HF$, then G will be in the curve; and if ZS = $\sqrt[3]{HB^3 + HZ^3} - HZ$, then S is in the curve. Or if the point S be given, then taking HB = $\sqrt[3]{HS^3 - HZ^3}$, then the point B is given, thro' which BN is to be drawn parallel to the horizon, by which the points G, g, &c. will be found.

2. Let ZFO be any curve, FH the radius of curvature at F; draw BN parallel to the horizon, and

Fig. and PD, Pd, Pn, parallel to HF, Hf, Hl. And
 132. if Dd express the weight of the arch stone FGrtf, then PD, Pd, will be the pressure of the joints FH, fH. Therefore by reasoning as before, if the pyramid PDn be equal to the arch stone FGrtf, then G will be in the curve. Whence if $PD^3 = HG^3 - HF^3$, or $HG = \sqrt[3]{PD^3 + HF^3}$, then G is in the curve. Therefore if the point S be given, take $PB = \sqrt[3]{HS^3 - HZ^3}$, HZ being the radius of curvature in Z. Then BN being drawn parallel to the horizon; from this all the points G will be found, as before mentioned.

E X A M. I.

133. Let ZO be a circle, $HZ = 1$, $ZS = \frac{1}{17}$, then $HB = .69$; then if $ZF = 30$ deg. then $FG = .14$. If $ZF = 60$ deg. then $FG = .55$, &c. for describing the curve SGG.

E X A M. 2.

134. Let ZFf be a parabola, $ZA = AO = 1$, $ZS = \frac{1}{8}$, M the focus, $ZM = \frac{1}{4}$, $Zm = \frac{1}{2}$, rad. curvature at $Z = \frac{1}{2}$, at $F = 1.41 = FH$, at $f = 2.6 = fb$. Then (PB or) $AB = \sqrt[3]{\frac{61}{512}} = .49$; then $HG = \sqrt[3]{AD^3 + HF^3} = 1.47$, and $FG = .06$; and $bg = \sqrt[3]{Ad^3 + bf^3} = 2.63$, and $fg = .03$; through which points, the curve SGG is to be described.

P R O B. LIX.

To find the fluent of $cx^2\dot{x} + y\dot{x} = a\dot{y}$.

This reduced is $\dot{y} = \frac{cx^2\dot{x}}{a} + \frac{y\dot{x}}{a}$, and the fluent is easily found by Sir J. Newton's method (Prob. II. Case II.) as follows.

$cx^2\dot{x}$

$$\begin{aligned}
 & \frac{cx^2x}{a} \\
 & + \frac{yx}{a} \quad + \frac{cx^3x}{3aa} + \frac{cx^4x}{4aa} + \frac{cx^5x}{4.5a^3} \\
 & y = \frac{cx^2x}{a} + \frac{cx^3x}{3aa} + \&c. \\
 & y = \frac{cx^3}{3a} + \frac{cx^4}{3.4aa} + \frac{cx^5}{3.4.5aa} + \frac{cx^6}{3.4.5.6a^3} \&c. \\
 & y = 2caa \times \frac{x^3}{2.3a^3} + \frac{x^4}{2.3.4a^4} + \frac{x^5}{2.3.4.5a^5} \&c. \\
 & = 2caa \times 1 + \frac{x}{a} + \frac{x^2}{2aa} + \frac{x^3}{2.3a^3} + \frac{x^4}{2.3.4a^4} + \frac{x^5}{2.3.4.5a^5} \&c. \\
 & - 2caa \times 1 + \frac{x}{a} + \frac{xx}{2aa} \\
 & = 2caa \times \text{numb. of hyp. log } \frac{x}{a} - 2caa - 2cax - cxx \\
 & = 2caa \times M^{\frac{x}{a}} - 2caa - 2cax - cxx.
 \end{aligned}$$

Where $M = 2.71828$, the number whose hypy. log. is 1.

PROB. LX.

To find the line of the first satellite of Jupiter's moving through the penumbra of Jupiter.

The apparent diameter of the sun is $32' 12''$ at the earth, and at Jupiter it is reciprocally as the distance. But Jupiter's mean distance is to the earth's mean distance from the sun, as 5.2 to 1. Therefore as $5.2 : 1 :: 32' 12''$ or $1932'' : 371.5''$; and this is the angle that the penumbra makes at Jupiter.

The periodic time of the first satellite is $1^d 18^h 27' 34''$ and that is $= 152854$ seconds. And the angle of the penumbra being 371.5 seconds.

There-

Fig. Therefore as 360 deg. or 1296000" the whole circumference : to 152854" the whole time of revolution : : so is 371".5 the angle of the penumbra : to 44" of time nearly. And therefore all the time the satellite stays in the penumbra, amounts only to 44 seconds of time.

Cor. The difference of time, between the satellite's appearing after its emerging out of Jupiter's shadow, when observed by the best and the worst telescopes, can never amount to 44 seconds; and very probably not to half as much.

This matter is easily tried with two telescopes of different goodness. But there may be as much difference in the goodness of eyes, as in the goodness of telescopes; and reason tells us not to apply bad ones of either sort.

F I N I S.



E R R A T A.

In page 430,

instead of

$$\begin{array}{l} 0 \left| \begin{array}{l} p \\ rA - 1 \end{array} \right| = A \\ 1 \left| \begin{array}{l} rA - 1 \\ rB - 2 \end{array} \right| = B \end{array}$$

read

$$\begin{array}{l} 0 \left| \begin{array}{l} p \\ rA - 1 \end{array} \right| = A \\ 1 \left| \begin{array}{l} rA - 1 \\ rB - 2 \end{array} \right| = B \end{array}$$

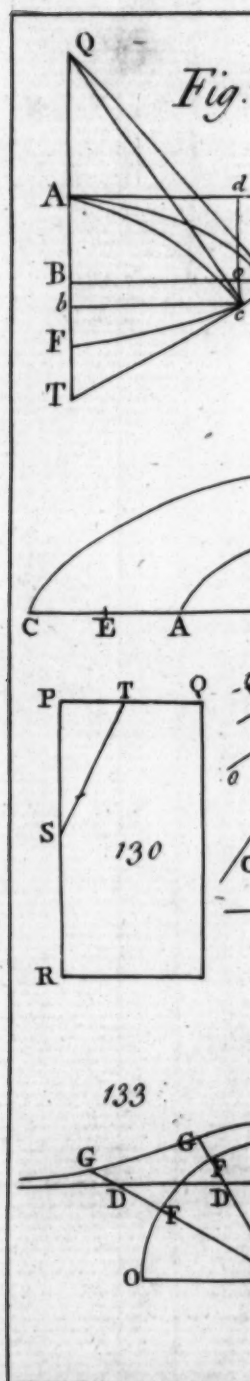


Fig. 125

